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BEAMS, PLATES AND SHELLS ON ELASTIC FOUNDATIONS

Translated from Russian

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FOREWORD

The theory of beams and plates on elastic foundations occupies a prominent place in contemporary structural mechanics. A very large number of studies have been devoted to this subject, and valuable practical methods for the analysis of beams and plates on elastic foundations have been worked out.

However, the existing calculation methods fall short of perfection, and leave unanswered many problems of practical importance. The large majority of these methods are too cumbrous for practical use; in addition, the assumptions made as regards the strains and stresses in natural soil cannot be fully ccepted. Complex three-dimensional structures on elastic foundations cannot be analyzed by existing methods. The hypothesis of a foundation modulus, by which the elastic foundation is considered as a system of separate unconnected springs, thus simplifying considerably the analysis of structures on elastic foundations, leads frequently to incorrect results.

On the other hand, by means of the hypothesis of an elastic isotropic semi-infinite space, we can describe correctly the physical properties of a natural foundation. This, however, leads to cumbrous calculations; as a result, practical solutions have been obtained only for a very restricted range of problems.

Establishing more accurate foundation models, and developing simplified methods for analyzing complex three-dimensional structures, taking into account the elasticity of the soil, are among the problems which the modern theory of structures on elastic foundations has to solve.

It can be expected that higher accuracy will be obtained by making allowance for the elastic-plastic deformation of the soil.

Approximate methods are obviously best suited for analyzing complex three-dimensional engineering structures on elastic foundations, since they lead to relatively simple expressions.

A new theory for analyzing structures on elastic foundations, based on Vlasov's general variational method, is proposed in the book. This theory is more accurate than the well-known theory of Winkler and Zimmermann, but is simpler than the theory of the elastic semi-infinite space.

This theory considers the elastic foundation (and, in general, the non-homogeneous foundation) as a single- or double-layer model whose properties are described by two or more generalized elastic characteristics. This model was proposed in 1949 by Vlasov in his book "Structural Mechanics of Thin-Walled Three-Dimensional Systems." The theory of the single-layer foundation was further developed by Leont'ev /11, 55/, Ruchkin /68/, Kosab'yan /45/, Cheche /81, 82/, etc.

The basic differential equation describing the state of strain of a loaded single-layer foundation has the same form as the solutions obtained by

Filonenko-Borodich /75, 76/ and Wieghardt. The models of Wieghardt and of Filonenko-Borodich are therefore mathematically equivalent to the single-layer model used here. An elastic-foundation model similar to the single-layer model was also considered by Pasternak /62/.

A merit of the theory proposed here is that the solution of many problems of practical importance is reduced to solving ordinary differential equations whose integrals can be found from tables. The simplicity of the mathematical methods and the clearness of the mathematical model make this theory very adaptable; not only the basic problems of beams and plates on elastic foundation, but also various more complex problems can be solved with its aid. These problems include the analysis of shells, taking into account additional transverse loads and the deformation of the underlying foundation, and problems of the dynamics and stability of structures on elastic foundations. The proposed theory can be applied to the determination of the stresses and strains in single- and multilayer strata of horizontal or inclined excavations.

The authors do not claim to have solved completely all problems of practical interest; nor do they consider that the methods proposed by them are universally applicable. Many problems are examined for the first time in this book, and, as a result, have not been worked out to the stage of formulas and tables for ready use. However, the extensive material, collected so far on the analysis of structures on elastic single-layer foundations, makes publication of such a book necessary. The authors hope that the book will be of use both for engineering practice and for further investigations.

The book consists of seven chapters. The first six chapters are devoted to problems of beams, plates, and spherical shells on elastic foundations, and to the dynamics and stability of such structures. Chapters I to III are mainly based on Leont'ev's thesis /55/; Chapters IV and V make use of the results of Ruchkin's studies /68/, kindly placed by him at the disposal of the authors. The last chapter (Chapter VII) describes a new approach to contact problems, based on the method of initial functions /10, 13, 14/, with whose aid complex three-dimensional problems of the theory of elasticity are reduced to two-dimensional problems; several examples illustrating this method are given. The bibliography indicates the main sources which were used by the authors in writing this book. The list given is, of course, incomplete. More complete bibliographies on the subject of structures on elastic foundations are given in /42, 50, and 64/.

This book is intended not only for research engineers, but also for engineers working in design and planning firms. Tables, dimensionless diagrams, and practical examples have been introduced in order to simplify practical calculations. The basic aim of the book is, however, to make available to the engineer an efficient variational method, with whose aid he himself will be able in each case to select a certain scheme of calculations, establish the corresponding model of the elastic foundation, and solve the problem by relatively simple mathematical means.

Chapters I, II, III, IV, V, and VI of this book were written by N. N. Leont'ev, and Chapter VII by V. Z. Vlasov. The editor was V. Z. Vlasov.

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V. Z. Vlasov, N. N. Leont'ev

Chapter I

APPLICATION OF THE GENERAL VARIATIONAL METHOD TO THE THEORY OF ELASTIC FOUNDATIONS

§ 1. FUNDAMENTALS OF THE VARIATIONAL METHOD USED IN REDUCING COMPLEX TWO-DIMENSIONAL PROBLEMS IN THE THEORY OF ELASTICITY TO ONE-DIMENSIONAL PROBLEMS

1

Consider a thin rectangular plate loaded by forces acting in its plane (Figure 1, a). Assume that the plate is deformed without bending, so that its state of stress is determined by normal stresses σ_x, σ_y and shearing stresses τ_{xy}, τ_{yx} only. The stresses σ_x, σ_y, τ_{xy}, τ_{yx} are independent of the coordinate z, it being assumed that the plate thickness δ is very small. In the theory of elasticity this is called a problem of plane stress.

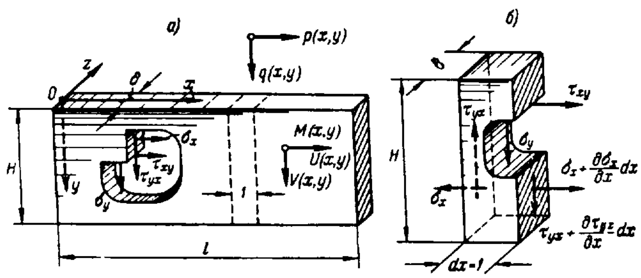


FIGURE 1.

Problems of plane stress are two-dimensional since the displacements, strains, and stresses are functions of the two coordinates x and y only. Two basic methods are available for solving such problems: the method of stresses and the method of displacements. The first method uses as basic unknowns the stresses $\sigma_x(x, y)$, $\sigma_y(x, y)$, $\tau_{xy}(x, y)$, $\tau_{yx}(x, y)$, which are determined from the conditions of continuity of the deformations. This method is similar to the method of forces used widely in the structural mechanics of statically indeterminate strut systems. The second method adopts as basic unknowns the displacements $u(x, y)$, $v(x, y)$, determined from the conditions of equilibrium of the elastic system. This method corresponds to the method of strains in structural mechanics.

We shall use here the method of displacements, adopting as principal unknowns the displacements $u(x, y)$ and $v(x, y)$ of a certain point $M(x, y)$ of the plate. The x-direction will be called longitudinal, and the y-direction, transverse. The displacements $u(x, y)$ and $v(x, y)$ will accordingly be called longitudinal and transverse displacements respectively. These displacements will be considered as positive if they are in the positive direction of the corresponding coordinate axis.

In the two-dimensional case, the stresses and strains are related as follows:

$$\left.\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy}), \\ \sigma_y &= \frac{E}{1-\nu^2}(\varepsilon_{yy} + \nu\varepsilon_{xx}), \\ \tau_{xy} &= \tau_{yx} = \frac{E}{2(1+\nu)}\varepsilon_{xy}, \end{aligned}\right\} \tag{1.1}$$

where E = modulus of elasticity, ν = Poisson's ratio for the material of the plate, $\varepsilon_{xx} = \varepsilon_{xx}(x, y)$ and $\varepsilon_{yy} = \varepsilon_{yy}(x, y)$ = strains in the longitudinal and transverse directions respectively, $\varepsilon_{xy} = \varepsilon_{xy}(x, y)$ = shearing strain.

The strain components ε_{xx}, ε_{yy}, ε_{xy}, are related to the unknown displacements u and v as follows:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{1.2}$$

The system (1.1) and (1.2) defines the states of stress and strain in the plate; when the displacements u and v are known, the problem can be considered solved.

2

In order to obtain a simple approximate solution, the unknown functions $u(x, y)$ and $v(x, y)$ are expanded in finite series:

$$\left.\begin{aligned} u(x, y) &= \sum_{i=1}^{m} U_i(x)\varphi_i(y) \qquad (i = 1, 2, 3, \ldots, m). \\ v(x, y) &= \sum_{k=1}^{n} V_k(x)\psi_k(y) \qquad (k = 1, 2, 3, \ldots, n). \end{aligned}\right\} \tag{1.3}$$

The functions $\varphi_i(y)$, $\psi_k(y)$ are assumed to be known, and the functions $U_i(x)$, $V_k(x)$to be unknown. It is often convenient to introduce dimensionless functions $\varphi_i(y)$, $\psi_k(y)$; the functions $U_i(x)$ and $V_k(x)$ will then have the dimension of length (displacement).

Because of the dimensions and physical meaning of expressions (1.3), the functions $U_i(x)$, $V_k(x)$ can be called generalized displacements. Indeed, each of the m functions $U_i(x)$, calculated for a given section $x = \text{const}$ of the plate, determines in a generalized form the magnitude of the longitudinal displacement $u_i(x, y)$ in this section. Similarly, each of the n functions $V_k(x)$ determines the magnitude of the transverse displacement $v_k(x, y)$ for the entire section $x = \text{const}$. The distributions of the longitudinal and transverse displacements over the sections $x = \text{const}$ are given respectively by the functions $\varphi_i(y)$ and $\psi_k(y)$, which are therefore called functions of the transverse distribution of the displacements.

Provided they are linearly independent and express the physical meaning of the problem, the functions $\varphi_i(y)$ and $\psi_k(y)$ approximating the state of strain in the plate in the transverse direction, can be chosen in different ways. Some examples will make this point clear. Consider the bending of a narrow plate (beam) with free lengthwise ends. Assume that the sections remain plane during bending and that no transverse elongation takes place. The unknown displacements $u(x, y)$ and $v(x, y)$ can be represented in this case in the form:

$$\left.\begin{aligned} u(x, y) &= U_1(x)\varphi_1(y) = U_1(x)\cdot y, \\ v(x, y) &= V_1(x)\psi_1(y) = V_1(x)\cdot 1 \end{aligned}\right\} \tag{1.4}$$

(the coordinate y is measured from the center of the cross section). The functions of the transverse distribution of the displacements are therefore in this case:

$$\varphi_1(y) = y, \quad \psi_1(y) = 1,$$

the remaining functions $\varphi_i(y)$ and $\psi_k(y)$ (i = 2 . . . m, k = 2 . . . n) are zero.

It follows from (1.4) that the generalized displacement $U_1(x)$ represents the angle of inclination of the section, and the generalized displacement $V_1(x)$, the plate deflection.

A second term can be added to the elementary solution (1.4), known from the theory of the strength of materials, for the bending of a narrow beam acted upon by a load antisymmetrical with respect to the x-axis (Figure 2);

$$\left.\begin{aligned} u(x, y) &= U_1(x)\, y + U_2(x)\sin\frac{2\pi y}{H}, \\ v(x, y) &= V_1(x)\, 1 + V_2(x)\cos\frac{\pi y}{H}. \end{aligned}\right\} \tag{1.5}$$

The following expressions have thus been selected for the functions $\varphi_i(y)$, $\psi_k(y)$:

$$\varphi_1(y) = y, \qquad \varphi_2(y) = \sin\frac{2\pi y}{H},$$
$$\psi_1(y) = 1, \qquad \psi_2(y) = \cos\frac{\pi y}{H}.$$

The first right-hand terms in (1.5) represent the displacements when the sections are assumed to remain plane; the second terms are introduced to correct the inaccuracies due to this assumption and that of zero transverse elongations.

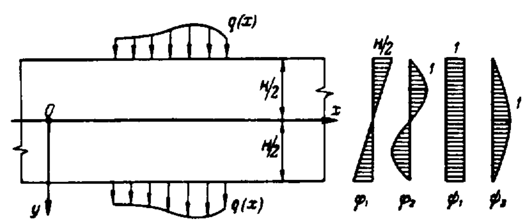


FIGURE 2.

A different procedure can be adopted to find a more accurate solution for a narrow beam. We imagine the beam to be divided into horizontal strips each of which is assumed to remain plane, although this assumption is not valid for the cross section as a whole. As an example, Figure 3 represents a plate with free upper edge and built-in lower edge (immovable both horizontally and vertically). This plate is divided into three parts along its height. It is assumed that the sections of each part remain plane, and that the transverse strains $\varepsilon_y = \frac{\partial v}{\partial y}$ are constant [over each section of the parts]. Equations (1.3) can then be written in the form:

$$\left.\begin{aligned} u(x, y) &= U_1(x)\varphi_1(y) + U_2(x)\varphi_2(y) + U_3(x)\varphi_3(y), \\ v(x, y) &= V_1(x)\psi_1(y) + V_2(x)\psi_2(y) + V_3(x)\psi_3(y). \end{aligned}\right\} \qquad (1.6)$$

The functions $\varphi_1(y)$, $\varphi_2(y), \ldots, \psi_2(y)$, $\psi_3(y)$ are represented in Figure 3. It is seen that in the range of variation of y, the functions of the transverse distribution of the displacements satisfy the continuity equations and the geometrical boundary conditions for $y = 0$ and $y = H$. The generalized displacement $U_1(x)$ determines the horizontal displacement on the plate surface, and the generalized displacement $V_1(x)$ equals the deflection of the upper edge of the plate. The remaining generalized displacements determine the displacements of the interior points of the plate along the lines $y = h_1$ and $y = h_2$.

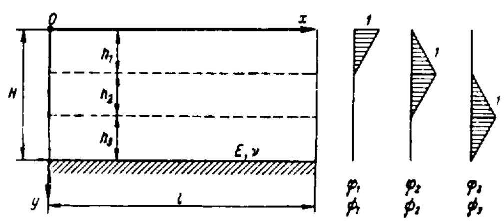


FIGURE 3.

The accuracy of the calculation increases with the number of parts into which the plate is divided (i.e. with the number of terms in (1.3)); the exact solution of the two-dimensional problem is obtained by passing to the limit $n \to \infty$ and $m \to \infty$.

The manner in which, in this example, the functions $\varphi_i(y)$ and $\psi_k(y)$ were chosen for a homogeneous isotropic thin plate may also be applied to a thin plate consisting of several horizontal layers having different elastic coefficients E and ν and thicknesses δ.

Depending on the problem and the accuracy required, the functions $\varphi_i(y)$, and $\psi_k(y)$ can be obtained as linearly independent and continuous functions of the coordinate y by many other methods also.

The representation of the unknown displacements by means of finite series (1.3) is equivalent to reducing the plate to a system having a finite number of degrees of freedom in the transverse direction and an infinite number of degrees of freedom in the longitudinal direction. Such systems can be called discrete-continuous, in contrast to the two-dimensional models of thin plates whose behavior is described by partial differential equations, in which the plates are considered as two-dimensional deformable solids possessing an infinite number of degrees of freedom in both the x- and y-directions.

It also follows from the series expansions (1.3) that the two-dimensional problem of the theory of elasticity has been reduced to a one-dimensional problem, since it suffices to determine m functions $U_i(x)$ and n functions $V_k(x)$ of the same variable in order to obtain the longitudinal and transverse displacements $u(x, y)$ and $v(x, y)$.

3.

The functions $U_i(x)$ and $V_k(x)$ can be obtained from the equilibrium conditions for an elementary strip of length $dx = 1$, delimited by the sections $x = \text{const}$ and $x + dx = \text{const}$ (Figure 1, b). In accordance with Lagrange's principle of virtual displacements, the equilibrium conditions are obtained by equating to zero the total work of all internal and external forces acting on this strip over any virtual displacement.

In accordance with (1.3), the virtual longitudinal displacements of the elementary strip are $\bar{u}_j = \varphi_j(y)$ for $U_j = 1$, where j can have m different values. The virtual transverse displacements of the strip are given in the form $\bar{v}_h = \psi_h(y)$ for $V_h = 1$, the subscript h denoting any of the n virtual displacements. Thus the vertical strip considered possesses $(m + n)$ degrees of freedom in the plane of the plate, m corresponding to longitudinal displacements (parallel to the x-axis), and n to transverse displacements (parallel to the y-axis).

The external forces acting on this strip are caused by the normal stresses σ_x, $\sigma_x + \frac{\partial \sigma_x}{\partial x} dx$, by the shearing stresses τ_{yx}, $\tau_{yx} + \frac{\partial \tau_{yx}}{\partial x} dx$, due to the interaction between the strip and the remainder of the plate, and by the given distributed load whose x and y components [per unit height] are $p(x, y)$ and $q(x, y)$ respectively. The internal forces acting in the strip are caused by the normal stresses σ_y and the shearing stresses τ_{xy}. The work done by all the external and internal forces of the strip over any of the $m + n$ virtual displacements is given by the following expressions:

$$\int \frac{\partial \sigma_x}{\partial x} \varphi_j \, dF - \int \tau_{xy} \varphi_j' \, dF + \int p(x, y) \varphi_j \, dy = 0 \quad (j = 1, 2, 3, \ldots, m), \tag{1.7}$$

$$\int \frac{\partial \tau_{yx}}{\partial x} \psi_h \, dF - \int \sigma_y \psi_h' \, dF + \int q(x, y) \psi_h \, dy = 0 \quad (h = 1, 2, 3, \ldots, n), \tag{1.8}$$

where $dF = \delta \, dy$ = element of plate cross section, δ = plate thickness.

In each equation (1.7) the total work done by all external and internal forces acting on the elementary strip in the longitudinal direction has been equated to zero. The first term represents the work of the external forces $\frac{\partial \sigma_x}{\partial x} dx dF$. The second term represents the work of the internal shearing forces $\tau_{xy} dF$. By virtue of (1.2) and (1.3), the shearing strains are given by the derivative $\varphi_j'(y)$ when $U_j = 1$. In each equation (1.8) the total work done by all forces acting on the elementary strip in the transverse direction has been equated to zero. As in (1.7), the first term represents the work of the external forces $\frac{\partial \tau_{yx}}{\partial x} dF$; the second term represents the work of the internal normal forces $\sigma_y dF$.

The last terms in equations (1.7) and (1.8) correspond to the virtual work done by the given loads.

The following expressions are obtained by inserting (1.2) and (1.3) into (1.1):

$$\left.\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}\Big[\sum_{i=1}^{m} U_i'\varphi_i + \nu\sum_{k=1}^{n} V_k\psi_k'\Big],\\
\sigma_y &= \frac{E}{1-\nu^2}\Big[\sum_{k=1}^{n} V_k\psi_k' + \nu\sum_{i=1}^{m} U_i'\varphi_i\Big],\\
\tau_{xy} = \tau_{yx} &= \frac{E}{2(1+\nu)}\Big[\sum_{i=1}^{m} U_i\varphi_i' + \sum_{k=1}^{n} V_k'\psi_k\Big].
\end{aligned}\right\} \tag{1.9}$$

Substitution of (1.9) in (1.7) and (1.8) leads to a system of ordinary differential equations in $U_i(x)$ and $V_k(x)$; this system consists of m equations corresponding to the m degrees of freedom of the strip in the longitudinal direction, and n equations corresponding to the n degrees of freedom of the strip in the transverse direction. This system can be written down as follows:

$$\left.\begin{aligned}
&\sum_{i=1}^{m} a_{ji}U_i'' - \frac{1-\nu}{2}\sum_{i=1}^{m} b_{ji}U_i + \sum_{k=1}^{n}\Big(\nu t_{jk} - \frac{1-\nu}{2}c_{jk}\Big)V_k' + \frac{1-\nu^2}{E}p_j = 0\\
&\qquad (j = 1, 2, 3, \ldots, m),\\
&-\sum_{i=1}^{m}\Big(\nu t_{hi} - \frac{1-\nu}{2}c_{hi}\Big)U_i' + \frac{1-\nu}{2}\sum_{k=1}^{n} r_{hk}V_k'' -\\
&\qquad - \sum_{k=1}^{n} s_{hk}V_k + \frac{1-\nu^2}{E}q_h = 0 \qquad (h = 1, 2, 3, \ldots, n).
\end{aligned}\right\} \tag{1.10}$$

When the functions $\varphi_i(y)$, $\varphi_j(y)\ (i, j = 1, 2, 3, \ldots, m)$, and $\psi_k(y)$, $\psi_h(y)$, $(k, h = 1, 2, 3, \ldots, n)$ have been chosen, and their derivatives are thus known, the coefficients in (1.10) are obtained from the following equations:

$$\left.\begin{aligned}
a_{ji} = a_{ij} &= \int \varphi_j\varphi_i\,dF, & r_{hk} = r_{kh} &= \int \psi_h\psi_k\,dF,\\
b_{ji} = b_{ij} &= \int \varphi_j'\varphi_i'\,dF, & s_{hk} = s_{kh} &= \int \psi_h'\psi_k'\,dF,\\
c_{jk} &= \int \varphi_j'\psi_k\,dF, & c_{hi} &= \int \psi_h\varphi_i'\,dF,\\
t_{jk} &= \int \varphi_j\psi_k'\,dF, & t_{hk} &= \int \psi_h'\varphi_k\,dF.
\end{aligned}\right\} \tag{1.11}$$

The integrals are taken over the entire width of the strip; in the general case δ can be a function of y.

The expressions (1.11) can be easily obtained from graphs of the functions $\varphi_i(y)$, $\psi_k(y)$, and their first derivatives.

If $p(x, y)$, and $q(x, y)$, are given the free terms $p_j = p_j(x)\ (j = 1, 2, 3, \ldots, m)$, and $q_h = q_h(x)\ (h = 1, 2, 3, \ldots, n)$ in (1.10) are obtained from the equations:

$$p_j = \int p(x, y)\varphi_j\,dy, \qquad q_h = \int q(x, y)\psi_h\,dy. \tag{1.12}$$

The loads $p(x, y)$ and $q(x, y)$ are considered positive when acting in the positive directions of the coordinate axes.

In the general case it is assumed that the loads $p(x, y)$ and $q(x, y)$ are distributed over the plate height as arbitrary functions of y. Expressions (1.12) may also apply to the external load acting at the longitudinal edges of the plate, which in the general case consists of given shearing and normal forces. In accordance with the physical meaning of these expressions (virtual work done by the loads), the concentrated forces must be included. Thus, if shearing and normal forces $p(x, 0)$, $q(x, 0)$ per unit length act at the upper edge of the plate in addition to the loads $p(x, y)$ and $q(x, y)$, we obtain for (1.12):

$$\left.\begin{aligned} p_j &= p(x)\varphi_j(0) + \int p(x, y)\varphi_j(y)\,dy, \\ q_h &= q(x)\psi_h(0) + \int q(x, y)\psi_h(y)\,dy. \end{aligned}\right\} \qquad (1.13)$$

Such integrals, extended both over distributed and concentrated loads, are called Stieltjes integrals.

The most efficient modern method of integrating a symmetrical system of ordinary differential equations with constant coefficients is Krylov's method, by which such a system can quickly be reduced to a single equivalent differential equation. In our case, the order of this equation will be $2(m+n)$. Hence, the unknown functions $U_i(x)$, $V_k(x)$ satisfying (1.10) will contain $2(m+n)$ arbitrary integration constants. The number of these constants is equal to the number of independent geometrical conditions to which the end sections $x=0$ and $x=l$ of the plate can be subjected (l = plate length in the longitudinal direction).

The position after deformation of all points of an arbitrary section $x = \text{const}$ is in fact defined by $m+n$ independent magnitudes: m functions $U_i(x)$ determine the positions of these points in the longitudinal direction (displacement from the plane $x = \text{const}$), and n functions $V_k(x)$ determine these positions along the height. Hence, $m+n$ magnitudes can be arbitrarily specified for one end section of the plate. The number of independent conditions for the two end sections $x=0$ and $x=l$ is thus $2(m+n)$, which is equal to the number of arbitrary integration constants. By varying these constants we can obtain a solution for the most varied geometrical boundary conditions in respect to the longitudinal and transverse displacements. Consider a plate with the boundary conditions at $x=0$ and $x=l$ given as stresses or, in the case of a mixed boundary problem, partially as displacements.

When the functions $\varphi_i(y)$ and $\psi_k(y)$ have been selected, the stresses σ_x and τ_{yx} at $x = \text{const}$ can be expressed through $m+n$ independent generalized statical magnitudes. The virtual work done by the normal and shearing forces $\sigma_x dF$ and $\tau_{yx} dF$ over any of the $m+n$ virtual displacements of the points of the section considered is:

$$\left.\begin{aligned} T_j(x) &= \int \sigma_x \varphi_j\,dF \qquad (j = 1, 2, 3, \ldots, m), \\ S_h(x) &= \int \tau_{yx}\psi_h\,dF \qquad (h = 1, 2, 3, \ldots, n), \end{aligned}\right\} \qquad (1.14)$$

where $dF = \delta\,dy$.

The integrals in (1.14) are taken over the entire cross section of the plate. The magnitudes $T_j(x)$ and $S_h(x)$ represent generalized longitudinal and transverse (shearing) forces acting in the section $x = \text{const}$ of the plate. Considering these magnitudes as internal forces, we can express them through the functions $U_i(x)$ and $V_k(x)$. It follows from (1.9), (1.11), and (1.14) that:

$$\left.\begin{aligned} T_j(x) &= E\Big(\sum_{i=1}^{m} a_{ji}U_i' + \nu \sum_{k=1}^{n} t_{jk}V_k\Big) \qquad (i, j = 1, 2, 3, \ldots, m), \\ S_h(x) &= G\Big(\sum_{i=1}^{m} c_{hi}U_i + \sum_{k=1}^{n} r_{hk}V_k'\Big) \qquad (h, k = 1, 2, 3, \ldots, n). \end{aligned}\right\} \tag{1.15}$$

Using (1.15), it becomes possible to impose $2(m+n)$ generalized boundary conditions expressed as stresses on the plate edges $x = 0$ and $x = l$.

Let a given system of distributed normal force $p^0(x_0, y)$ and distributed shearing forces $q^0(x_0, y)$ act at edge $x = x_0$ (Figure 4).

We imagine an elementary strip dx to be cut from the plate; from the principle of virtual displacements, we obtain the following equilibrium conditions:

$$\left.\begin{aligned} &\int (\sigma_x^0\delta - p^0)\varphi_j\,dy = 0 \qquad (j = 1, 2, 3, \ldots, m), \\ &\int (\tau_{yx}^0\delta - q^0)\psi_h\,dy = 0 \qquad (h = 1, 2, 3, \ldots, n). \end{aligned}\right\} \tag{1.16}$$

Inserting (1.14) into (1.16) yields:

$$S_h^0(x) = \int q^0\psi_h\,dy. \quad T_j^0(x) = \int p^0\varphi_j\,dy, \tag{1.17}$$

We have thus obtained the relationship between the generalized forces (1.15) and the specified external loads at $x = x_0$.

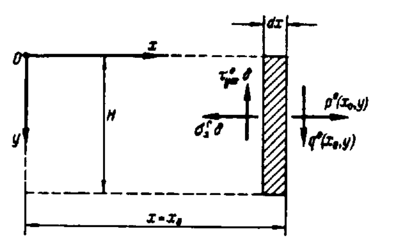


FIGURE 4.

After obtaining the general integral of (1.10), it is possible, with the aid of (1.17) and (1.15), to determine the strains and stresses in the plate for any boundary conditions at $x = 0$ and $x = l$, expressed as stresses, displacements, or both.

§ 2. TWO-DIMENSIONAL DEFORMATIONS OF ELASTIC FOUNDATIONS. CALCULATION MODELS

Consider now the inverse two-dimensional problem. Let the elastic foundation be a compressible layer of thickness H placed on a rigid foundation (Figure 5). The dimensions of this compressible layer in the z-direction are assumed to be large. We also assume that the external load is independent of the z coordinate and acts in planes parallel to the xy plane. The thickness of the elastic foundation, its support conditions, the elastic constants and all the other conditions are constant in the z-direction. In the theory of elasticity this is called plane strain, since the displacements of all points occur in planes perpendicular to the z-axis.

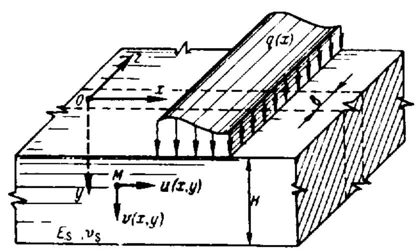


FIGURE 5.

We imagine a narrow plate of thickness δ to be cut from the elastic foundation by two planes parallel to the xy plane (Figure 5). The stresses σ_x, σ_y, τ_{xy}, τ_{yx}, the strain components $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$, and the displacements u and v of this plate are functions of x and y only, and are, as in plane stress, related by (1.1) and (1.2). In the case of plane strain, the following have to be substituted for E and ν in (1.1):

$$E_0 = \frac{E_s}{1-\nu_s^2}, \qquad \nu_0 = \frac{\nu_s}{1-\nu_s}, \tag{2.1}$$

where E_s and ν_s are the modulus of elasticity and Poisson's ratio for the foundation material (soil) respectively.

In order to determine the strains and stresses in the plate by the method of displacements, we express $u(x, y)$ and $v(x, y)$ by expansions (1.3). As in section 1, we obtain the following system of $(m+n)$ ordinary differential equations in $U_i(x)$ and $V_k(x)$ from the conditions of equilibrium of an elementary strip of width $dx = 1$ (Figure 1):

$$\left.\begin{aligned}
&\sum_{i=1}^{m} a_{ji}U_i'' - \frac{1-\nu_0}{2}\sum_{i=1}^{m} b_{ji}U_i + \sum_{k=1}^{n}\left(\nu_0 t_{jk} - \frac{1-\nu_0}{2}c_{jk}\right)V_k' + \\
&\qquad\qquad + \frac{1-\nu_0^2}{E_0}p_j = 0 \qquad (j = 1, 2, 3, \ldots, m), \\
&-\sum_{i=1}^{m}\left(\nu_0 t_{hi} - \frac{1-\nu_0}{2}c_{hi}\right)U_i' + \frac{1-\nu_0}{2}\sum_{k=1}^{n} r_{hk}V_k'' - \sum_{k=1}^{n} s_{hk}V_k + \\
&\qquad\qquad + \frac{1-\nu_0^2}{E_0}q_h = 0 \qquad (h = 1, 2, 3, \ldots, n).
\end{aligned}\right\} \tag{2.2}$$

The system (2.2) differs from (1.10) only by the elastic constants. The coefficients a_{li}, b_{li}, ..., c_{hi}, t_{hi} are again determined from (1.11) and depend only on the functions $\varphi_i(y)$ and $\psi_k(y)$.

As before, the free terms p_l and q_h represent the work done over the displacements $\varphi_l(y)$ and $\psi_h(y)$ respectively, by the given horizontal and vertical distributed loads $p(x, y)$ and $q(x, y)$, and are obtained in the general case from (1.13).

The volume forces distributed over the foundation are usually neglected when the deformation of an elastic foundation is considered; only surface forces (the loads applied to the foundation surface) are taken into consideration. The free terms in (2.2) are in this case:

$$\left.\begin{aligned} p_l &= p(x)\cdot\varphi_l(0), \\ q_h &= q(x)\cdot\psi_h(0), \end{aligned}\right\} \tag{2.3}$$

where $p(x)$ and $q(x)$ are the shearing and normal surface forces respectively; $\varphi_l(0)$ and $\psi_h(0)$ are the values of the functions $\varphi_l(y)$ and $\psi_h(y)$ at the foundation surface $y = 0$.

After the functions $U_i(x)$ and $V_k(x)$ have been determined from (2.2) and the corresponding boundary conditions, the displacements of the elastic foundation $u(x, y)$ and $v(x, y)$ can be found from (1.3), and the stresses σ_x, σ_y, τ_{xy} from (1.9); the elastic constants are given by (2.1).

The system of ordinary differential equations (2.2) thus defines the plane strain of an elastic foundation considered as a linearly deformable medium of finite thickness H. Because of the limited number of terms in expansions (1.3) the solution obtained will be an approximation of the exact solution of the theory of elasticity. At the same time, the system (2.2) can be considered as defining a generalized model of the elastic foundation, based on the general variational method. Different models can be obtained by selecting different expressions for the functions $\varphi_i(y)$ and $\psi_k(y)$. Although only approximations from the point of view of the theory of elasticity, these models are nevertheless sufficiently accurate for practical application. Their accuracy can be increased at will by increasing the number of terms in (1.3).

Increasing the number of terms in (1.3) is, however, undesirable, since an increase in the order of the differential equations (2.2) results. The accuracy of the solution can also be increased by a better selection of the functions $\varphi_i(y)$ and $\psi_k(y)$. Since this selection is based on experimental data or on a more rigorous theoretical analysis, a sufficiently accurate solution can be obtained even with a minimum number of terms in (1.3).

Consider for example an elastic foundation in which the horizontal displacements are either zero or negligible. In this case:

$$\left.\begin{aligned} u(x, y) &= 0, \\ v(x, y) &= \sum_{k=1}^{n} V_k(x)\psi_k(y). \end{aligned}\right\} \tag{2.4}$$

System (2.2) then becomes:

$$\frac{1-\nu_0}{2}\sum_{k=1}^{n} r_{hk}V_k'' - \sum_{k=1}^{n} s_{hk}V_k + \frac{1-\nu_0^2}{E_0}q_h = 0, \tag{2.5}$$

where

$$r_{hk} = \int \psi_h \psi_k \, dF, \qquad s_{hk} = \int \psi'_h \psi'_k \, dF.$$

In this model the load may spread due to cohesion: reactions may appear even outside the region of load application. The model described by (2.5) can be represented schematically as a system of elementary elastic columns (springs) mutually interacting as a result of internal friction and adhesion (Figure 6).

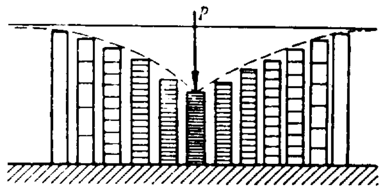


FIGURE 6.

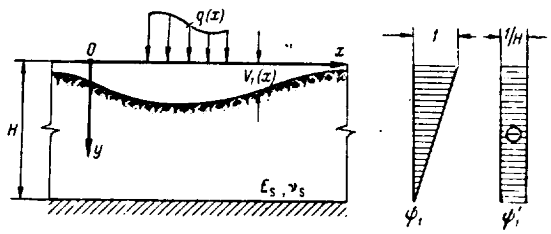


FIGURE 7.

The properties of this model depend on the functions $\psi_k(y)$ and on the number of terms in (2.4). Since this is a particular case of the generalized model described by (2.2), we can obtain from it even simpler models of the elastic foundation by the introduction of additional hypotheses. Assuming, for example, that the elastic foundation forms a thin compressible layer whose base is fixed, we can write:

$$v(x, y) = V_1(x)\psi_1(y), \tag{2.6}$$

$$\psi_1(y) = \frac{H-y}{H}. \tag{2.7}$$

The function $V_1(x)$ thus represents the settling of the foundation surface (Figure 7).

From (2.6) and (2.7) we obtain for (2.5) the single differential equation:

$$\frac{1-\nu_0}{2} r_{11} V_1'' - s_{11} V_1 + \frac{1-\nu_0^2}{E_0} q_1 = 0, \tag{2.8}$$

where

$$\left.\begin{aligned} r_{11} &= \int_0^H \psi_1^2 \, dF = \frac{\delta H}{3}, \\ s_{11} &= \int_0^H \psi_1'^2 \, dF = \frac{\delta}{H}. \end{aligned}\right\} \tag{2.9}$$

Assuming that the external distributed load $q(x)$ is applied only to the foundation surface, the term q_1 given by (2.3) will be:

$$q_1 = q(x).$$

The model described by (2.8) can be called a model with two characteristics, or simply a single-layer model (/11/).

The stresses in this model are obtained approximately by substituting (2.4), (2.6), and (2.7) in (1.9):

$$\left.\begin{aligned} \sigma_y &= \frac{E_0}{(1-\nu_0^2)} V_1\psi_1' = -\frac{E_0}{(1-\nu_0^2)H} V_1(x), \\ \tau_{yx} = \tau_{xy} &= \frac{E_0}{2(1+\nu_0)} V_1'\psi_1 = \frac{E_0}{2(1+\nu_0)} \frac{H-y}{H} V_1'(x). \end{aligned}\right\} \tag{2.10}$$

The normal stresses σ_y thus remain constant over the height of the foundation, while the shearing stresses τ_{yx} vary linearly.

If the thickness H of the compressible layer is large, the behavior of the elastic foundation will be described only approximately by (2.7) since in this case σ_y cannot be assumed to remain constant over the height. In order to increase the accuracy without increasing the number of terms in (2.6), it is necessary to select for ψ_1 an expression more closely describing the actual decrease of the displacements and stresses with depth. We may, for instance, write (Figure 8)

$$\psi_1 = \frac{\operatorname{sh}\gamma(H-y)}{\operatorname{sh}\gamma H}, \tag{2.11}$$

where γ is a constant determining the rate of decrease of the displacements with depth. In this case the solution is given by (2.8), but the coefficients r_{11} and s_{11} are found from (2.11).

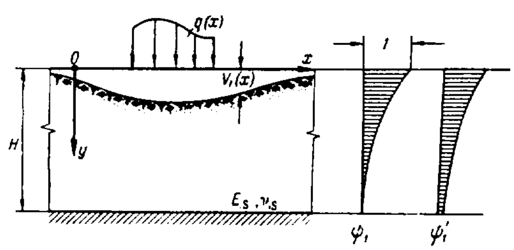


FIGURE 8.

If horizontal displacements in the foundation cannot be neglected, and if the foundation is sufficiently thin and fixed to its base, we can write:

$$u(x, y) = U_1\varphi_1, \quad v(x, y) = V_1\psi_1,$$

where

$$\varphi_1 = \frac{H-y}{H}, \quad \psi_1 = \frac{H-y}{H}. \tag{2.12}$$

System (2.2) then becomes:

$$\left.\begin{aligned} a_{11}U_1'' - \frac{1-\nu_0}{2} b_{11}U_1 + \left(\nu_0 t_{11} - \frac{1-\nu_0}{2} c_{11}\right) V_1' + \frac{1-\nu_0^2}{E_0} p_1 = 0, \\ -\left(\nu_0 t_{11} - \frac{1-\nu_0}{2} c_{11}\right) U_1' - \frac{1-\nu_0}{2} r_{11} V_1 - s_{11} V_1 + \frac{1-\nu_0^2}{E_0} q_1 = 0. \end{aligned}\right\} \tag{2.13}$$

The coefficients in (2.13) can be found by inserting (2.12) into (1.11). We can also write:

$$\varphi_1 = \frac{\operatorname{sh}\gamma(H-y)}{\operatorname{sh}\gamma H}, \quad \psi_1 = \frac{\operatorname{sh}\gamma(H-y)}{\operatorname{sh}\gamma H} \tag{2.14}$$

or other suitable expressions.

If the foundation consists of several horizontal layers having different elastic properties, the functions $\varphi_i(y)$, $\psi_k(y)$ can be selected as in section 1 (Figure 3). The modulus of elasticity can then be assumed to vary over the height. This model of the elastic foundation is thus called a multilayer model. A multilayer model can be used for a homogeneous elastic foundation when the thickness H is considerable; the solution obtained is far more accurate than that obtained from the single-layer model described by (2.8) or (2.13).

By selecting the functions $\varphi_i(y)$ and $\psi_k(y)$ differently, we obtain from (2.2) an infinity of different models of the elastic foundation describing with sufficient accuracy the peculiarities of the problem under consideration. Since the selection of the correct model of the elastic foundation is very important in the design of structures resting on such foundations, the advantages of the general variational method are obvious.

Most models obtained by this method are simpler than the model of an elastic semi-infinite plane based on the methods of Zhemochkin and Gorbunov-Posadov. Henceforth only the simplest, i.e., the single-layer model, will be considered. This simple model makes better allowance for the elastic properties of the soil than the well-known model of Winkler and Zimmermann, while permitting the design of beams, plates, and more intricate structures resting on elastic foundations by simple mathematical methods.

§ 3. PLANE MODEL OF THE ELASTIC FOUNDATION WITH TWO CHARACTERISTICS

1. Basic differential relationships

Let the elastic foundation be a compressible layer of thickness H (Figure 9). Assume that the displacements in this layer due to the surface load, are approximately:

$$u(x, y) = 0, \quad v(x, y) = V_1(x)\psi_1(y), \tag{3.1}$$

where $\psi_1(y)$ is a function of y, selected according to the nature of the problem.

According to (1.2) and (3.1) the strain components are as follows:

$$\left.\begin{aligned} \varepsilon_{yy} &= V_1(x)\psi_1'(y), \\ \varepsilon_{xy} &= V_1'(x)\psi_1(y), \\ \varepsilon_{xx} &= 0, \end{aligned}\right\} \tag{3.2}$$

The normal and shearing stresses are obtained from (1.9):

$$\left.\begin{aligned}\sigma_y &= \frac{E_0}{(1-\nu_0^2)} V_1(x)\,\psi_1'(y),\\ \tau_{yx} &= \frac{E_0}{2(1+\nu_0)} V_1'(x)\,\psi_1(y).\end{aligned}\right\} \tag{3.3}$$

The constants E_0 and ν_0 are as follows [cf. (2.1)]:

$$E_0 = \frac{E_s}{1-\nu_s^2}, \qquad \nu_0 = \frac{\nu_s}{1-\nu_s}, \tag{3.4}$$

where E_s and ν_s are the modulus of elasticity and Poisson's ratio respectively for the material of the foundation.

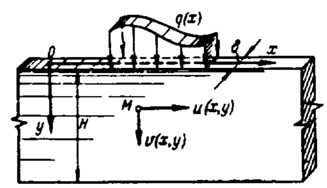


FIGURE 9.

The system (2.2) is in this case reduced to a single equation [cf. (2.8)] containing the only given function of y, $\psi_1(y)$:

$$\frac{1-\nu_0}{2} r_{11} V_1'' - s_{11} V_1 + \frac{1-\nu_0^2}{E_0} q_1 = 0. \tag{3.5}$$

The free term in (3.5) represents the work done by the distributed surface load $q(x)$ and is:

$$q_1(x) = q(x)\,\psi_1(0). \tag{3.6}$$

The coefficients in (3.5) are:

$$\left.\begin{aligned}r_{11} &= \int_0^H \psi_1^2(y)\,dF,\\ s_{11} &= \int_0^H \psi_1'^2(y)\,dF,\end{aligned}\right\} \tag{3.7}$$

where $dF = \delta\,dy$.

After multiplying each term by $\frac{E_0}{1-\nu_0^2}$, (3.5) can be written:

$$2tV_1'' - kV_1 + q_1 = 0, \tag{3.8}$$

where

$$\left.\begin{aligned}k &= \frac{E_0 s_{11}}{1-\nu_0^2}\\ t &= \frac{E_0 r_{11}}{4(1+\nu_0)}.\end{aligned}\right\} \tag{3.9}$$

Differential equation (3.8) relates the vertical displacements of the foundation to the load applied at its surface. It differs from the well-known relationship obtained by assuming a direct proportionality (foundation-modulus hypothesis) by the presence of a term containing the second derivative of the generalized displacement V_1. This term, multiplied by 2t, makes allowance for the shearing stresses in the elastic foundation.

This model of the elastic foundation thus differs basically from the Winkler-Fuss model*. Since allowance is made for the shearing stresses, the load can spread, i.e., displacements occur not only directly beneath the load, but also at other points (Figure 6).

The properties of the elastic foundation satisfying (3.8) are defined by the two integral characteristics (3.9). The characteristic k determines the compressive strain in the elastic foundation; it is thus similar to the foundation modulus. The characteristic t determines the shearing strain in the elastic foundation; it thus defines the load-spreading capacity of the foundation**.

The solution of (3.8) requires the establishment of boundary conditions; these should be given in integral form, either as generalized forces or as generalized displacements.

From (3.1) and (3.3), we obtain:

$$\left.\begin{aligned} T_i &= \int \sigma_x \varphi_i \, dF = 0, \\ S_1 &= \int \tau_{yx} \psi_1 \, dF = \frac{E_0 b}{2(1+\nu_0)} V_1' \int \psi_1^2 \, dy = 2tV_1'. \end{aligned}\right\} \tag{3.10}$$

2. Selecting the function of the transverse distribution of the displacements

The distribution of the displacements and normal stresses over the height H of the elastic foundation, and thus the basic properties of this foundation, are determined by the function $\psi_1(y)$. In the previous section the following function was chosen for a sufficiently thin compressible layer, throughout which the normal stresses σ_y are constant:

$$\psi_1(y) = \frac{H-y}{H}. \tag{3.11}$$

The foundation is assumed to be fixed on its base (Figure 7). The strain in the y direction is constant:

$$\varepsilon_{yy} = -V_1(x)\frac{1}{H};$$

the normal stresses are also constant over the layer height:

$$\sigma_y = -\frac{E_0}{H(1-\nu_0^2)} V_1(x). \tag{3.12}$$

* The hypothesis of the foundation modulus, usually called "Winkler hypothesis", was first proposed by the Russian academician Fuss in 1801.

** Pasternak proposed to call the characteristics k and t "the two foundation moduli"; the single-layer model would thus be called "the model with two foundation moduli" /62/. Equation (3.8) is identical with the solution obtained by Filonenko-Borodich for his simplest model of the elastic foundation /76/, and also with Wieghardt's solution.

We then obtain for the coefficients (3.7) [cf. (2.9)]:

$$\left.\begin{aligned} r_{11} &= \int_0^H \psi_1^2\, dF = \frac{\delta H}{3}, \\ s_{11} &= \int_0^H \psi_1'^2\, dF = \frac{\delta}{H}, \end{aligned}\right\} \tag{3.13}$$

the constants in (3.9) become:

$$\left.\begin{aligned} k &= \frac{E_0\delta}{H(1-\nu_0^2)}, \\ t &= \frac{E_0\delta H}{12(1+\nu_0)}. \end{aligned}\right\} \tag{3.14}$$

From (3.10) we obtain:

$$S_1 = \frac{E_0\delta H}{6(1+\nu_0)} V_1'(x) = 2tV_1'(x). \tag{3.15}$$

Expressions (3.14) and (3.11) are valid also for elastic foundations of considerable thickness, consisting of several compressible layers having different elastic properties, and in particular for a semi-infinite elastic plane. In this case, H in (3.11) and (3.14) defines the height of an equivalent layer throughout which the normal stresses σ_y are assumed to be constant (Figure 10). This height can be determined by comparing the displacements of the foundation surface, given by (3.8), with the actual displacements.

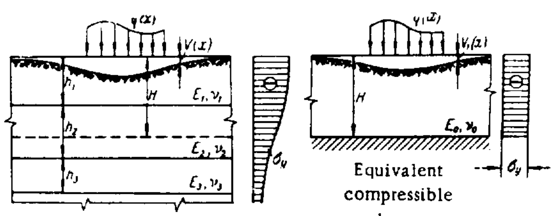


FIGURE 10.

If the concept of equivalent layer is undesirable, we can choose the following expression for $\psi_1(y)$ when the elastic foundation is deep:

$$\psi_1(y) = \frac{\operatorname{sh}\gamma(H-y)}{\operatorname{sh}\gamma H}, \tag{3.16}$$

where H is the depth of the subsoil (for a semi-infinite elastic plane $H \to \infty$), and γ is a coefficient depending on the elastic properties of the foundation and determining the rate of decrease of the displacements over the depth of the foundation.

In accordance with (3.16), the normal stresses in the foundation are:

$$\sigma_y = -\frac{E_0\gamma}{(1-\nu_0^2)} V_1(x) \frac{\operatorname{ch}\gamma(H-y)}{\operatorname{sh}\gamma H}, \tag{3.17}$$

and the elastic constants in (3.8) become:

$$k=\frac{E_0\delta}{H(1-\nu_0^2)}\psi_k,\quad t=\frac{E_0\delta H}{12(1+\nu_0)}\psi_t, \tag{3.18}$$

where

$$\left.\begin{aligned}\psi_k&=\frac{\gamma H}{2}\,\frac{\operatorname{sh}\gamma H\operatorname{ch}\gamma H+\gamma H}{\operatorname{sh}^2\gamma H},\\ \psi_t&=\frac{3}{2}\,\frac{1}{\gamma H}\,\frac{\operatorname{sh}\gamma H\operatorname{ch}\gamma H-\gamma H}{\operatorname{sh}^2\gamma H}\end{aligned}\right\} \tag{3.19}$$

We again obtain:

$$S_1=\int_0^H\tau_{yx}\psi_1\,dF=2tV_1'(x), \tag{3.20}$$

where t is given by (3.18).

In this case the normal stresses σ_y are not constant, but vary as the hyperbolic cosine (Figures 8 and 14). The characteristics (3.18) define the elastic properties of the soil more accurately than the characteristics (3.14).

When $H\to\infty$ the characteristics (3.14) tend toward infinitely and zero respectively, while the characteristics (3.18) remain finite. Hence, (3.16) and (3.18) are valid even when the thickness of the elastic layer becomes infinite; expression (3.16) can be used for the approximate calculation of structures on a semi-infinite elastic plane.

Similar results can be obtained when the function $\psi_1(y)$ is an exponential function:

$$\psi_1(y)=e^{-\gamma y}, \tag{3.21}$$

which also adequately describes the decrease of the displacements and stresses over the depth of the elastic foundation.

Depending on the nature of the problem, many analytical expressions in addition to (3.11), (3.16), and (3.21) can be selected, either based on experimental data or on solutions obtained by the methods of the theory of elasticity.

3. Action of a concentrated vertical force.

We shall determine the displacements of an elastic foundation, due to a concentrated force P acting at the origin of coordinates (Figure 11). In this case we obtain from (3.8) the following homogeneous differential equation for the displacements $V_1(x)$:

$$2tV_1''-k_1V_1=0. \tag{3.22}$$

The coefficients are obtained from (3.9) and (3.7).

The general integral of (3.22) is:

$$V_1(x)=C_1e^{-\alpha x}+C_2e^{\alpha x}, \tag{3.23}$$

where

$$\alpha = \sqrt{\frac{k}{2t}}. \qquad (3.24)$$

For reasons of symmetry we need consider only the right-hand half of the foundation. One of the integration constants in (3.23) is determined from the condition that the foundation displacement at infinity must be zero:

at $x \to \infty$ $$V_1(x) \to 0. \qquad (3.25)$$

Hence

$$C_2 = 0.$$

The second integration constant is found from the conditions at $x = 0$.

We can define the generalized shearing force $S_1(x)$ as the work done by all forces acting at the section $x = \text{const}$ over the virtual displacements $\bar{v}_1(x, y) = 1 \cdot \psi_1(y)$ when $V_1(x) = 1$. It has a discontinuity at those sections where concentrated forces act on the elastic-foundation surface (Figure 11).

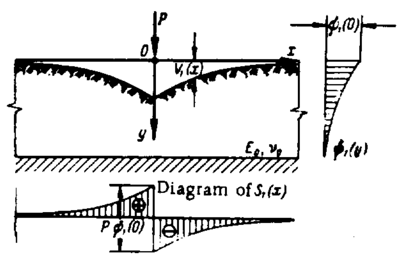


FIGURE 11.

Taking into account the symmetry of the problem we find [from (3.10)] that:

at $x = 0$ $$S_1(0) = -\frac{P}{2}\psi_1(0), \qquad (3.26)$$

where $\psi_1(0)$ is the value of $\psi_1(y)$ at the foundation surface.

From (3.10) and (3.26) we obtain:

$$2\alpha t C_1 = \frac{P}{2}\psi_1(0),$$

whence

$$C_1 = \frac{P}{4}\frac{\psi_1(0)}{\alpha t}. \qquad (3.27)$$

The displacement of any point of the elastic foundation can now be written:

$$v(x, y) = P\frac{\psi_1(0)}{4\alpha t}e^{-\alpha x}\psi_1(y). \qquad (3.28)$$

where

$$\left.\begin{array}{lll}\alpha = \sqrt{\dfrac{k}{2t}}, & k = \dfrac{E_0 s_{11}}{1 - \nu_0^2}, & t = \dfrac{E_0 r_{11}}{4(1 + \nu_0)}, \\ r_{11} = \delta \int\limits_0^H \psi_1^2(y)\,dy, & & s_{11} = \delta \int\limits_0^H \psi_1'^2(y)\,dy. \end{array}\right\} \tag{3.29}$$

If the linear expression (3.11) is selected for the function of the transverse distribution of the displacements, (3.28) becomes:

$$v(x, y) = \frac{3(1 - \nu_0^2)}{\sqrt{6(1 - \nu_0)}} \frac{P}{E_0 \delta} e^{-\alpha x} \frac{H - y}{H}, \tag{3.30}$$

where

$$\alpha = \sqrt{\frac{k}{2t}} = \frac{1}{H} \frac{\sqrt{6(1 - \nu_0)}}{(1 - \nu_0)}.$$

As an example, Figure 12 shows the dimensionless displacements $\bar{V}(x)$ of the foundation surface as a function of x/H, obtained from (3.30) for $\nu_0 = 0$.

The actual displacements of the foundation surface are:

$$V_1(x) = \frac{P}{E_0 \delta} \bar{V}(x).$$

It is seen that the displacements decrease rapidly with increasing distance from the point of load application.

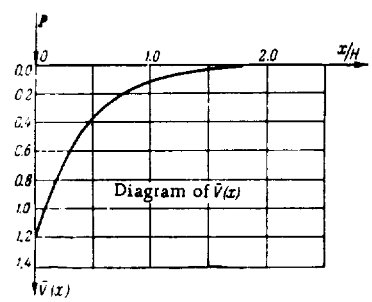


FIGURE 12.

When the function $\psi_1(y)$ is given by (3.16), expression (3.28) becomes:

$$v(x, y) = \frac{3(1 - \nu_0^2)}{\sqrt{6(1 - \nu_0)}} \frac{1}{\psi_t \psi_\alpha} \frac{P}{E_0 \delta} e^{-\alpha x} \frac{\operatorname{sh} \gamma (H - y)}{\operatorname{sh} \gamma H}, \tag{3.31}$$

where

$$\left.\begin{array}{l}\alpha = \sqrt{\dfrac{k}{2t}} = \dfrac{1}{H} \dfrac{\sqrt{6(1 - \nu_0)}}{(1 - \nu_0)} \psi_\alpha, \\ \psi_t = \dfrac{3}{2} \dfrac{1}{\gamma H} \dfrac{\operatorname{sh} \gamma H \operatorname{ch} \gamma H - \gamma H}{\operatorname{sh}^2 \gamma H}, \\ \psi_\alpha = \gamma H \sqrt{\dfrac{1}{3} \dfrac{\operatorname{sh} \gamma H \operatorname{ch} \gamma H + \gamma H}{\operatorname{sh} \gamma H \operatorname{ch} \gamma H - \gamma H}}. \end{array}\right\} \tag{3.32}$$

[cf. (3.18), (3.19)]

We can write (3.31) in the form:

$$v(x, y) = \frac{P}{E_0\delta} \bar{V}(x) \psi_1(y), \tag{3.33}$$

and plot diagrams of the dimensionless displacement

$$\bar{V}(x) = \frac{3(1-\nu_0^2)}{\sqrt{6(1-\nu_0)}} \frac{1}{\psi_1\psi_a} e^{-ax} \tag{3.34}$$

for different values of the parameter $\bar{\gamma} = \gamma H$. Such curves are drawn in Figure 13 for $\bar{\gamma} = 1$, $\bar{\gamma} = 2$, $\bar{\gamma} = 1$ (for $\nu_0 = 0$). Figure 14 shows the function $\psi_1(y)$, plotted for the same values of $\bar{\gamma}$, and also the distribution of the normal stresses σ_y over the foundation height, obtained from (3.17).

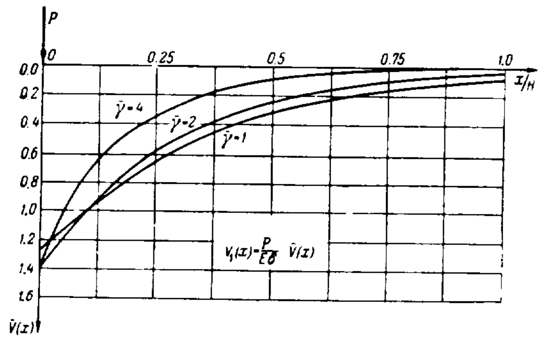


FIGURE 13.

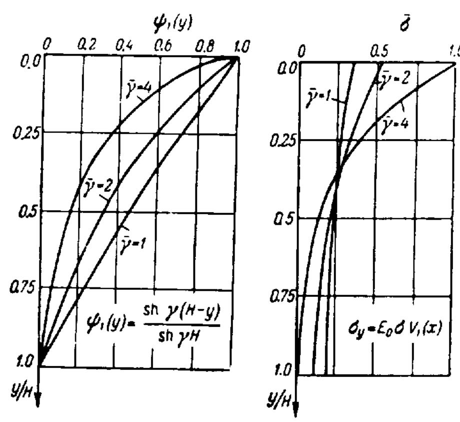


FIGURE 14.

It is seen that an increase in the parameter $\bar{\gamma}$ causes the displacements and normal stresses to decrease more rapidly with increasing depth.

The normal-stress diagrams also show that the proposed model of the elastic foundation is to a certain degree artificial: it gives finite (nonzero) values for the normal stresses at points on the foundation surface which carry no load. This is a result of employing the variational method, which applies the equilibrium conditions in integral form without providing for their fulfilment at every single point of the system.

Since the subject of this book is the analysis of structures on elastic foundations, and not the stresses in the elastic foundation itself, these shortcomings may be ignored.

4. Case of a distributed load

The displacements of an elastic foundation, due to a load $q(x)$ distributed over its surface are best obtained from (3.28), which for $y=0$ determines the displacements of the elastic foundation, due to a concentrated force P. If we put $P=1$, the curve of displacements becomes an influence line and can be used to determine the displacements of any point of the surface at any load.

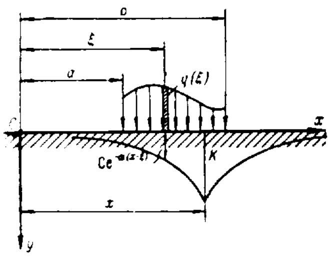


FIGURE 15.

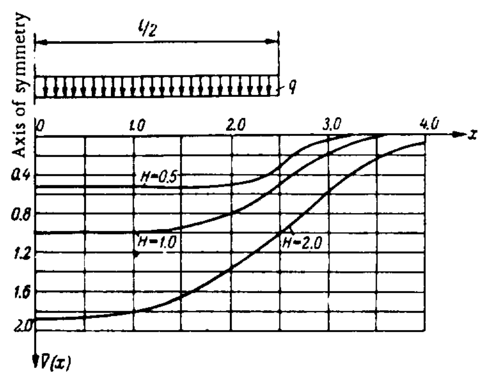


FIGURE 16.

If the applied load $q(\xi)$ is a known function of the distance ξ from the coordinate origin, we obtain for the foundation displacements at point K (Figure 15) [whose coordinates are $(x, 0)$]:

at $a \leqslant x \leqslant b$

$$V_1(x) = C_1\left[\int_a^x q(\xi)\, e^{-\alpha(x-\xi)}\, d\xi + \int_x^b q(\xi)\, e^{\alpha(x-\xi)}\, d\xi\right]; \qquad (3.35)$$

at $x > b$

$$V_1(x) = C_1 \int_a^b q(\xi)\, e^{-\alpha(x-\xi)}\, d\xi; \qquad (3.36)$$

at $x < a$

$$V_1(x) = C_1 \int_a^b q(\xi)\, e^{\alpha(x-\xi)}\, d\xi. \qquad (3.37)$$

where

$$C_1 = \frac{\psi_1(0)}{4\alpha t}. \tag{3.38}$$

In the particular case of a uniformly distributed load q, (3.35) becomes: at $a < x < b$

$$V_1(x) = \frac{qC_1}{\alpha}\left[2 - e^{-\alpha(x-a)} - e^{\alpha(x-b)}\right]. \tag{3.39}$$

We assume that the displacement decreases linearly with increasing depth:

$$\psi_1 = \frac{H-y}{H},$$

the constant C_1 is then found from (3.14):

$$C_1 = \frac{3(1+\nu_0)}{\alpha E \delta H},$$

and (3.39) becomes:

$$V_1(x) = \frac{q}{2k}\left[2 - e^{-\alpha(x-a)} - e^{\alpha(x-b)}\right], \tag{3.40}$$

where

$$k = \frac{E\delta}{H(1-\nu_0^2)}.$$

Figure 16 shows the dimensionless displacements $\bar{V}(x)$ obtained from (3.40) for several values of H (for $\nu_0 = 0$). The actual displacements are:

$$V_1(x) = \frac{q}{E_0\delta}\bar{V}(x).$$

It is seen that with decreasing H the behavior of the foundation approaches that of the Winkler model. With increasing H the displacement curve becomes smoother, and the absolute values of the displacements increase.

§ 4. SINGLE-LAYER FOUNDATION WITH VARIABLE ELASTIC PROPERTIES

1

The determination of the strains and stresses of an elastic foundation subjected to a load becomes considerably more difficult when the elastic properties of the foundation vary.

Consider an elastic foundation whose thickness H varies linearly in the x-direction (Figure 17).

We shall express the displacement of a point $M(x, y)$ of the foundation as before:

$$u(x, y) = 0, \quad v(x, y) = V_1(x)\psi_1(x, y). \tag{4.1}$$

It will be assumed that the function $\psi_1(x, y)$ varies linearly:

$$\psi_1(x, y) = \frac{H-y}{H}, \tag{4.2}$$

where

$$H = H_0 - \theta_0 x, \tag{4.3}$$

and θ_0 = tg β_0 (Figure 17).

The condition of equilibrium of an elementary strip of width $dx = 1$, cut from the foundation, is derived from (1.7), (1.8), [(1.9)], (4.1), (4.2), and (4.3). We obtain:

$$\frac{1-\nu_0}{6}\delta H(x) V'' - \frac{1-\nu_0}{6}\delta\theta_0 V' - \frac{\delta}{H(x)}\left[1 + \frac{\theta_0^2(1-\nu_0)}{6}\right]V + \frac{1-\nu_0^2}{E_0}q = 0. \tag{4.4}$$

According to (4.1), the generalized shearing force is:

$$S = \int_0^H \tau_{yx}\psi_1 dF. \left(V_1(x) = 1\right) \tag{4.5}$$

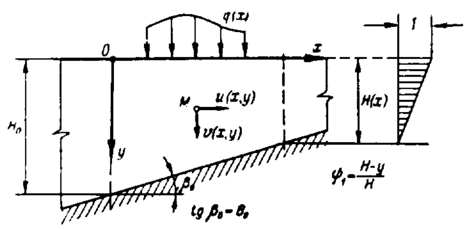


FIGURE 17.

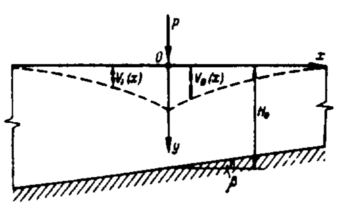


FIGURE 18.

Inserting (4.1) and (4.2) into the last expression of (1.1) yields:

$$\tau_{yx} = \frac{E_0}{2(1+\nu_0)}\frac{\partial v}{\partial x} = \frac{E_0}{2(1+\nu_0)}\left[V_1'\frac{H-y}{H} - \frac{\theta_0 y}{H^2}V_1\right], \tag{4.6}$$

where $H = H_0 - \theta_0 x$.

Substituting (4.6) and (4.2) in (4.5) and integrating, we find:

$$S_1 = \frac{E_0\delta}{12(1+\nu_0)}[2H(x)V_1' - \theta_0 V_1]. \tag{4.7}$$

2

Consider the particular case of a concentrated force P acting on the surface of the elastic foundation at the origin of coordinates (Figure 18). The following homogeneous differential equation is obtained:

$$\frac{1-\nu_0}{6}H(x)V_1'' - \frac{1-\nu_0}{6}\theta_0 V_1' - \frac{1}{H(x)}\left[1 + \frac{\theta_0^2(1-\nu_0)}{6}\right]V_1 = 0. \tag{4.8}$$

This is an Eulerian differential equation and can be written in the following form after cancellation and multiplying each term by $\frac{6H(x)}{1-\nu_0}$:

$$H^2(x)V_1'' + nH(x)V_1' + mV_1 = 0, \qquad (4.9)$$

where

$$n = -\theta_0, \quad m = -\left(\theta_0^2 + \frac{6}{1-\nu_0}\right).$$

Substituting $V = H^\lambda$ in (4.9) we obtain an equation with constant coefficients

$$V_1'' + (n-1)V_1' + mV_1 = 0. \qquad (4.10)$$

The roots of the auxiliary equation

$$\lambda^2 + (n-1)\lambda + m = 0$$

are

$$\lambda_{1,2} = -\frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} - m}. \qquad (4.11)$$

Since m must be negative, both roots (4.11) are real:

$$\lambda_1 = -r_1, \quad \lambda_2 = r_2. \qquad (4.12)$$

The general integral of (4.8) is:

$$V_1 = C_1(H_0 - \theta_0 x)^{-r_1} + C_2(H_0 - \theta_0 x)^{r_2}. \qquad (4.13)$$

Since the displacements of the foundation [at $x = \frac{H_0}{\theta_0}$ and] at infinity are zero, the constant C_2 must be zero for $x < 0$ while the constant C_1 must be zero for $x > 0$. Hence:

$$V_I = C_1(H_0 - \theta_0 x)^{-r}, \qquad V_{II} = C_2(H_0 - \theta_0 x)^{r_2}, \qquad (4.14)$$

where V_I and V_{II} are the displacements of the surface of the elastic foundation to the left and to the right, respectively, of the point where the force acts (Figure 18).

To determine the constants of integration, we note that:

at $x = 0$
$$V_I = V_{II}, \quad S_I - S_{II} = P, \qquad (4.15)$$

where S_I and S_{II} are the generalized shearing forces to the left and right, respectively, of $x = 0$.

Substituting (4.14) in (4.7) yields:

$$S_I = \frac{-E_0\delta(2r_1+1)\theta_0}{12(1+\nu_0)H^{r_1}}C_1, \qquad S_{II} = \frac{E_0\delta(2r_2-1)H^{r_2}\theta_0}{12(1+\nu_0)}C_2. \qquad (4.16)$$

By inserting (4.14) and (4.16) into (4.15) we obtain the integration constants:

$$C_1 = -\frac{6(1+\nu_0)H_0^{r_1}}{E_0\delta\theta_0(r_1+r_2)}P, \qquad C_2 = -\frac{6(1+\nu_0)H_0^{-r_2}}{E_0\delta\theta_0(r_1+r_2)}P. \qquad (4.17)$$

Lastly, by inserting (4.2), (4.14), and (4.17) into (4.1) we find the vertical displacement of any point of the elastic foundation:

at $x < 0$

$$-v(x, y) = \frac{6(1+\nu_0) H_0^{r_1}}{E_0 \delta \theta_0 (r_1 + r_2)} P H^{-r_1}(x) \frac{H(x) - y}{H(x)};$$

at $x > 0$

$$-v(x, y) = \frac{6(1+\nu_0) H_0^{-r_2}}{E_0 \delta \theta_0 (r_1 + r_2)} P H^{r_2}(x) \frac{H(x) - y}{H(x)},$$

where r_1 and r_2 are determined by (4.11).

§ 5. DOUBLE-LAYER ELASTIC FOUNDATION

1

Consider an elastic foundation of thickness $H = h_1 + h_2$, undergoing plane deformations (Figure 19). The two layers have different moduli of elasticity and Poisson ratios.

In accordance with (1.3), the displacements of a point of the elastic foundation are given by the following expressions:

$$\left.\begin{aligned} &u(x, y) = 0, \\ &v(x, y) = V_1(x)\psi_1(y) + V_2(x)\psi_2(y), \end{aligned}\right\} \tag{5.1}$$

where $\psi_1(y)$ and $\psi_2(y)$ are the functions of the transverse distribution of the displacements, and $V_1(x)$, $V_2(x)$ are the generalized vertical displacements.

The functions $\psi_1(y)$, $\psi_2(y)$ are chosen according to the nature of the problem. In particular, expressions (3.11), (3.16), or (3.21) can be used.

If the upper layer is thin and the lower layer thick, we can write (Figure 20):

$$\left.\begin{aligned} &\text{at } 0 \leqslant y \leqslant h_1 \qquad \psi_1 = \frac{h_1 - y}{h_1}, \quad \psi_2 = \frac{y}{h_1}; \\ &\text{at } h_1 \leqslant y \leqslant H \qquad \psi_1 = 0, \quad \psi_2 = \frac{\operatorname{sh}\gamma(H - y)}{\operatorname{sh}\gamma h_2}, \end{aligned}\right\} \tag{5.2}$$

where γ is a coefficient determining the rate of decrease of the displacements with depth.

In this case, the generalized displacements $V_1(x)$ and $V_2(x)$ define the vertical displacements respectively of the surface of the elastic foundation and of the boundary between the two layers.

From (2.2), (5.1), and (5.2) we obtain the following two differential equations for the determination of the functions $V_1(x)$ and $V_2(x)$:

$$\left.\begin{aligned} &\frac{1-\nu_1}{2}(r_{11}V_1'' + r_{12}V_2'') - (s_{11}V_1 + s_{12}V_2) + \frac{1-\nu_1^2}{E_1} q = 0, \\ &\frac{E_1}{2(1+\nu_1)} r_{12}V_1'' + \left[\frac{E_1}{2(1+\nu_1)} r_{22} + \frac{E_2}{2(1+\nu_2)} r_{22}'\right] V_2'' - \frac{E_1}{1-\nu_1^2} s_{12} V_1 - \\ &\qquad - \left[\frac{E_1}{1-\nu_1^2} s_{22} + \frac{E_2}{1-\nu_2^2} s_{22}'\right] V_2 = 0, \end{aligned}\right\} \tag{5.3}$$

where

$$\left.\begin{aligned}
r_{11} &= \int_0^{h_1} \psi_1^2\,dF = \frac{\delta h_1}{3}, & s_{11} &= \int_0^{h_1} \psi_1'^2\,dF = \frac{\delta}{h_1},\\
r_{12} &= \int_0^{h_1} \psi_1\psi_2\,dF = \frac{\delta h_1}{6}, & s_{12} &= \int_0^{h_1} \psi_1'\psi_2'\,dF = -\frac{\delta}{h_1},\\
r_{22} &= \int_0^{h_1} \psi_2^2\,dF = \frac{\delta h_1}{3}, & s_{22} &= \int_0^{h_1} \psi_2'^2\,dF = \frac{\delta}{h_1},\\
r_{22} &= \int_{h_1}^{H} \psi_2^2\,dF = \frac{\delta h_2}{3}\psi_t, & s_{22}' &= \int_{h_1}^{H} \psi_2'^2\,dF = \frac{\delta}{h_2}\psi_k.
\end{aligned}\right\} \tag{5.4}$$

The elastic constants E_1, E_2, ν_1 and ν_2, entering in (5.3) define the properties of the elastic foundation in plane strain. For a soil block, these are:

$$\left.\begin{aligned}
E_1 &= \frac{E_{1.s}}{1-\nu_{1.s}^2}, & \nu_1 &= \frac{\nu_{1.s}}{1-\nu_{1.s}},\\
E_2 &= \frac{E_{2.s}}{1-\nu_{2.s}^2}, & \nu_2 &= \frac{\nu_{2.s}}{1-\nu_{2.s}},
\end{aligned}\right\} \tag{5.5}$$

where $E_{1.s}, E_{2.s}, \nu_{1.s}, \nu_{2.s}$, are the moduli of elasticity and Poisson ratios of the first and second layers respectively.

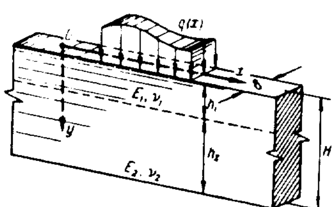


FIGURE 19.

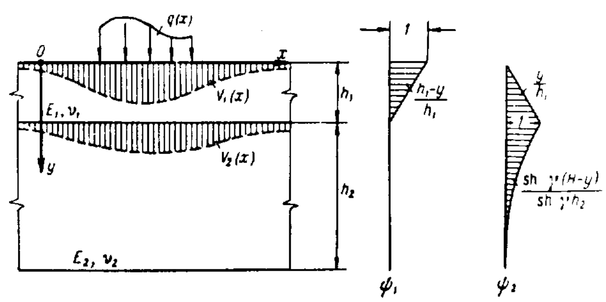


FIGURE 20.

Substituting (5.4) in (5.3), we obtain:

$$\left.\begin{aligned}
2t_1V_1'' - k_1V_1 + t_1V_2'' + k_1V_2 + q &= 0,\\
t_1V_1'' + k_1V_1 + 2(t_1+t_2)V_2'' - (k_1+k_2)V_2 &= 0,
\end{aligned}\right\} \tag{5.6}$$

where

$$k_1 = \frac{E_1\delta}{h_1(1-\nu_1^2)}, \qquad t_1 = \frac{E_1h_1\delta}{12(1+\nu_1)}, \tag{5.7}$$

$$k_2 = \frac{E_2\delta}{h_2(1-\nu_2^2)}\psi_k, \qquad t_2 = \frac{E_2h_2\delta}{12(1+\nu_2)}\psi_t, \tag{5.8}$$

and ψ_k and ψ_t are known from (3.19).

The coefficients k_1 and k_2 determine the compressive strains of the upper and lower layers respectively, while the coefficients t_1 and t_2 define their shearing strains.

In order to solve system (5.6), we introduce a function $F(x)$. The displacements $V_1(x)$ and $V_2(x)$ are then expressed through $F(x)$ and its derivatives in such a way that when these expressions are inserted into the second equation (5.6), the latter becomes an identity. The expressions which satisfy this condition are:

$$\left.\begin{aligned} V_1(x) &= (k_1 + k_2)F(x) - 2(t_1 + t_2)F''(x), \\ V_2(x) &= k_1F(x) + t_1F''(x). \end{aligned}\right\} \tag{5.9}$$

Substituting these expressions in the first equation of (5.6) yields:

$$t_1(3t_1 + 4t_2)F^{IV} - 2(3t_1k_1 + t_1k_2 + t_2k_1)F'' + k_1k_2F = q(x). \tag{5.10}$$

Differential equation (5.10) defines the stresses and strains in a double-layer elastic foundation. In order to solve specific problems it is necessary to add to this equation the relevant boundary conditions which are given in a generalized form in this method. We therefore introduce generalized internal forces corresponding to the generalized displacements $V_1(x)$ and $V_2(x)$. Since an elementary transverse strip cut from the foundation possesses two degrees of freedom in its plane, it follows from (1.14) that:

$$\left.\begin{aligned} S_1 &= \int_0^H \tau_{yx}\psi_1\,dF, \\ S_2 &= \int_0^H \tau_{yx}\psi_2\,dF, \end{aligned}\right\} \tag{5.11}$$

where $dF = \delta\,dy$.

By (5.1) and (5.2) the shearing stresses τ_{yx} are:

at $0 \leqslant y \leqslant h_1$

$$\tau_{yx} = \frac{E_1}{2(1+\nu_1)}\left(V_1'\frac{h_1 - y}{h_1} + V_2'\frac{y}{h_1}\right);$$

at $h_1 \leqslant y \leqslant H$

$$\tau_{yx} = \frac{E_2}{2(1+\nu_2)}V_2'\frac{\operatorname{sh}\gamma(H-y)}{\operatorname{sh}\gamma h_2}. \tag{5.12}$$

Substituting (5.12) in (5.11) and integrating over the entire height of the elastic foundation, we obtain:

$$\left.\begin{aligned} S_1 &= t_1(2V_1' + V_2'), \\ S_2 &= t_1V_1' + 2(t_1 + t_2)V_2'. \end{aligned}\right\} \tag{5.13}$$

Using (5.9), S_1 and S_2 can be expressed through $F(x)$:

$$\left.\begin{aligned} S_1 &= t_1[-(3t_1 + 4t_2)F''' + (3k_1 + 2k_2)F'], \\ S_2 &= (3t_1k_1 + t_1k_2 + 2t_2k_1)F'. \end{aligned}\right\} \tag{5.14}$$

The double-layer model can be called a foundation with four elastic characteristics. It permits a higher accuracy than the single-layer model characterized by only two independent parameters k and t (cf. section 3)

Different schemes of the elastic foundation can be obtained according to the selection of the parameters k_1, k_2, t_1, and t_2. Only one such scheme will be considered.

It is seen from (5.7) that when both h_1 and E_1 decrease, t_1 tends toward zero while k_1 remains finite. If we assume that a thin compressible soil layer near the surface of the elastic foundation has a modulus of elasticity considerably smaller than the lower layers, we can write for the first layer

$$t_1 = 0, \quad k_1 = K, \tag{5.15}$$

where K is a coefficient analogous to the foundation modulus and depends on the properties of the elastic foundation near the surface.

The double-layer foundation thus consists of an upper layer subject only to compressive stresses, $(t_1 = 0)$, similar in this sense to the Winkler foundation, while the lower layer is subject to both compressive and shearing stresses.

Substituting (5.15) in (5.9), (5.13), and (5.14) yields:

$$V_1 = (K + k_2) F - 2t_2 F'', \quad V_2 = KF; \tag{5.16}$$

$$S_1 = 0, \quad S_2 = S = 2t_2 V_2' = 2Kt_2 F'. \tag{5.17}$$

Inserting (5.15) into (5.10) yields:

$$-2t_2 F'' + k_2 F = \frac{q(x)}{K}. \tag{5.18}$$

Unlike (5.10), (5.18) can be applied to a double-layer foundation with upper Winkler layer. The term "double-layer foundation" will henceforth be applied only to this particular case of a double-layer model.

3

Let a concentrated force P act on the elastic foundation at the origin of coordinates (Figure 21). The following homogeneous differential equation is then obtained for the determination of $F(x)$:

$$-2t_2 F'' + k_2 F = 0. \tag{5.19}$$

Equation (5.19) is identical with (3.22). The following solution is obtained by analogy with (3.31):

$$F(x) = \frac{3(1-\nu_2^2)}{\sqrt{6(1-\nu_2)}} \frac{1}{\psi_t \psi_\alpha} \frac{P}{E_2 \delta K} e^{-\alpha_2 x}, \tag{5.20}$$

where

$$\alpha_2 = \sqrt{\frac{k_2}{2t_2}} = \frac{1}{h_2} \frac{\sqrt{6(1-\nu_2)}}{1-\nu_2} \psi_\alpha, \tag{5.21}$$

while ψ_t and ψ_α are given by (3.32).

Substitution of (5.20) in the second equation (5.16) yields:

$$V_2 = \frac{3(1-\nu_2^2)}{\sqrt{6(1-\nu_2)}} \frac{1}{\psi_t \psi_\alpha} \frac{P}{E_2 \delta} e^{-\alpha_2 x}. \tag{5.22}$$

which is identical with (3.31). By substituting (5.20) in the first equation (5.16) it is seen that, except at the point where the force acts:

$$V_1 = V_2$$

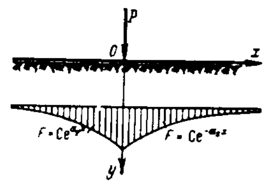


FIGURE 21.

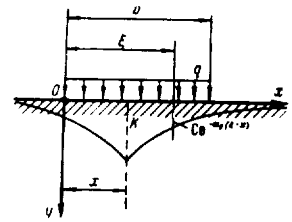


FIGURE 22.

4

When a uniformly distributed load q acts on a double-layer foundation (Figure 22), it is easiest to obtain the solution by using function (5.20), which for $P = 1$ represents an influence line.

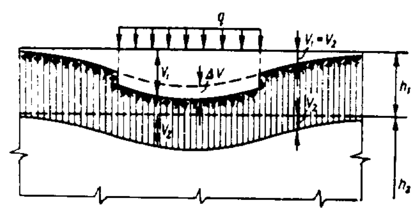


FIGURE 23.

By analogy with (3.36) and (3.40) we find:

$$\left.\begin{aligned} &\text{at } 0 \leqslant x \leqslant b \qquad F = \frac{q}{2Kk_2}\left[2 - e^{-\alpha_2 x} - e^{\alpha_2 (x-b)}\right]; \\ &\text{at } x > b \qquad F = \frac{q}{2Kk_2}\left[e^{-\alpha_2 (x-b)} - e^{-\alpha_2 x}\right]. \end{aligned}\right\} \tag{5.23}$$

Substituting (5.23) in (5.16) yields:

$$\text{at } 0 \leqslant x \leqslant b \qquad \left.\begin{aligned} V_1 &= \frac{q(K + k_2)}{Kk_2} - \frac{q}{2k_2}\left(e^{-\alpha_2 x} + e^{\alpha_2 (x-b)}\right), \\ V_2 &= \frac{q}{k_2} - \frac{q}{2k_2}\left(e^{-\alpha_2 x} + e^{\alpha_2 (x-b)}\right); \end{aligned}\right\} \tag{5.24}$$

$$\text{at } x > b \qquad V_1 = V_2 = \frac{q}{2k_2}\left(e^{\alpha_2 (x-b)} - e^{-\alpha_2 x}\right). \tag{5.25}$$

It is seen from (5.24) and (5.25) that the surface layer exhibits no strains outside the zone of load application (this corresponds to the postulation of the foundation modulus), since here $V_1 = V_2$. Within the zone of load application, we have:

$$\Delta V = V_1 - V_2 = \frac{q}{K}. \tag{5.26}$$

The function V_2 is everywhere continuous, while V_1 has a discontinuity at the borders of the zone of load application (Figure 23).

§ 6. THREE-DIMENSIONAL DEFORMATIONS OF AN ELASTIC FOUNDATION

Consider now a three-dimensional elastic foundation of thickness H placed above an incompressible layer (Figure 24). Let an external load, whose x, y z components are respectively $p(x, y, z)$, $g(x, y, z)$, and $q(x, y, z)$, act on this foundation. As in the two-dimensional problem, we shall use the method of displacements to determine the stresses and strains in the elastic foundation. The unknowns will be the displacements $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$ of a point $M(x, y, z)$ of the foundation. The displacements will be considered positive when their directions coincide with the positive directions of the corresponding coordinate axes.

By analogy with the two-dimensional problem, the unknown displacements u, v, w are represented by the following expansions:

$$\left.\begin{aligned} u(x, y, z) &= \sum_{i=1}^{m} u_i(x, y)\varphi_i(z) \qquad (i = 1, 2, 3, \ldots, m), \\ v(x, y, z) &= \sum_{g=1}^{l} v_g(x, y)\varkappa_g(z) \qquad (g = 1, 2, 3, \ldots, l), \\ w(x, y, z) &= \sum_{k=1}^{n} w_k(x, y)\psi_k(z) \qquad (k = 1, 2, 3, \ldots, n). \end{aligned}\right\} \tag{6.1}$$

The functions $\varphi_i(z)$, $\varkappa_g(z)$, $\psi_k(z)$ in (6.1) determine the variation with height of the horizontal and vertical displacements. They are assumed to be known dimensionless, linearly independent functions. The functions $u_i(x, y)$, $v_g(x, y)$, $w_k(x, y)$, which have the dimensions of length, are the unknowns. In accordance with their physical meaning, they will be called generalized displacements.

The normal and shearing stresses in the elastic foundation are in the three-dimensional case:

$$\left.\begin{aligned} \sigma_x &= \frac{E_0}{1-\nu_0^2}\left[\frac{\partial u}{\partial x} + \nu_0\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right], \\ \sigma_y &= \frac{E_0}{1-\nu_0^2}\left[\frac{\partial v}{\partial y} + \nu_0\left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x}\right)\right], \\ \sigma_z &= \frac{E_0}{1-\nu_0^2}\left[\frac{\partial w}{\partial z} + \nu_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right], \\ \tau_{zy} = \tau_{yz} &= \frac{E_0}{2(1+\nu_0)}\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right], \\ \tau_{zx} = \tau_{xz} &= \frac{E_0}{2(1+\nu_0)}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right], \\ \tau_{xy} = \tau_{yx} &= \frac{E_0}{2(1+\nu_0)}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]. \end{aligned}\right\} \tag{6.2}$$

where

$$E_0 = \frac{E_s}{1-\nu_s^2}, \qquad \nu_0 = \frac{\nu_s}{1-\nu_s}, \tag{6.3}$$

E_s and ν_s are respectively modulus of elasticity and Poisson's ratio for the elastic foundation.

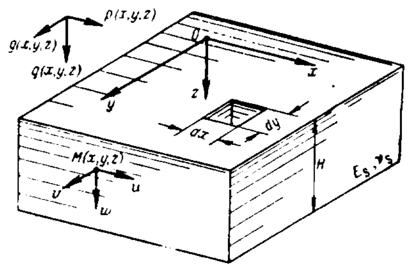


FIGURE 24.

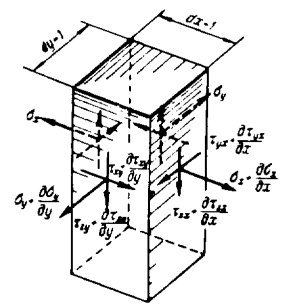


FIGURE 25.

Substitution of (6.1) in (6.2) gives the following expressions for the stresses as functions of the generalized displacements:

$$\left.\begin{aligned}
\sigma_x &= \frac{E_0}{1-\nu_0^2}\left[\sum_{i=1}^{m}\frac{\partial u_i}{\partial x}\varphi_i + \nu_0\left(\sum_{g=1}^{l}\frac{\partial v_g}{\partial y}\varkappa_g + \sum_{k=1}^{n} w_k\psi_k'\right)\right],\\
\sigma_y &= \frac{E_0}{1-\nu_0^2}\left[\sum_{g=1}^{l}\frac{\partial v_g}{\partial y}\varkappa_g + \nu_0\left(\sum_{k=1}^{n} w_k\psi_k' + \sum_{i=1}^{m}\frac{\partial u_i}{\partial x}\varphi_i\right)\right],\\
\sigma_z &= \frac{E_0}{1-\nu_0^2}\left[\sum_{k=1}^{n} w_k\psi_k' + \nu_0\left(\sum_{i=1}^{m}\frac{\partial u_i}{\partial x}\varphi_i + \sum_{g=1}^{l}\frac{\partial v_g}{\partial y}\varkappa_g\right)\right].
\end{aligned}\right\} \tag{6.4}$$

$$\left.\begin{aligned}
\tau_{zy} = \tau_{yz} &= \frac{E_0}{2(1+\nu_0)}\left[\sum_{k=1}^{n}\frac{\partial w_k}{\partial y}\psi_k + \sum_{g=1}^{l} v_g\varkappa_g'\right],\\
\tau_{zx} = \tau_{xz} &= \frac{E_0}{2(1+\nu_0)}\left[\sum_{i=1}^{m} u_i\varphi_i' + \sum_{k=1}^{n}\frac{\partial w_k}{\partial x}\psi_k\right],\\
\tau_{xy} = \tau_{yx} &= \frac{E_0}{2(1+\nu_0)}\left[\sum_{g=1}^{l}\frac{\partial v_g}{\partial x}\varkappa_g + \sum_{i=1}^{m}\frac{\partial u_i}{\partial y}\varphi_i\right].
\end{aligned}\right\} \tag{6.5}$$

In order to determine the functions $u_i(x, y)$, $v_g(x, y)$, $w_k(x, y)$, we cut from the foundation an elementary column of height H and sides $dx = 1$, $dy = 1$, (Figures 24 and 25). This column possesses $(m + l + n)$ degrees of freedom in the three directions. The generalized equilibrium conditions of the elementary column (considered as virtual displacements) can therefore be written in the form:

$$\left.\begin{aligned}
&\int \frac{\partial \sigma_x}{\partial x} \varphi_j \, dz - \int \tau_{xz} \varphi_j' \, dz + \int \frac{\partial \tau_{xy}}{\partial y} \varphi_j \, dz + \int p \varphi_j \, dz = 0 \\
&\qquad (j = 1, 2, 3, \ldots, m), \\
&\int \frac{\partial \sigma_y}{\partial y} \varkappa_f \, dz - \int \tau_{yz} \varkappa_f' \, dz + \int \frac{\partial \tau_{yx}}{\partial x} \varkappa_f \, dz + \int q \varkappa_f \, dz = 0 \\
&\qquad (f = 1, 2, 3, \ldots, l), \\
&\int \frac{\partial \tau_{zx}}{\partial x} \psi_h \, dz - \int \sigma_z \psi_h' \, dz + \int \frac{\partial \tau_{zy}}{\partial y} \psi_h \, dz + \int q \psi_h \, dz = 0 \\
&\qquad (h = 1, 2, 3, \ldots, n).
\end{aligned}\right\} \quad (6.6)$$

[cf. (1.7), (1.8)]

Each equation (6.6) states that the total work done by all external and internal forces acting on the elementary column over the corresponding virtual displacement equals zero:

$$\bar{u}_j(x, y, z) = \varphi_j(z),$$
$$\bar{v}_f(x, y, z) = \varkappa_f(z),$$
$$\bar{w}_h(x, y, z) = \psi_h(z)$$

for

$$u_j(x, y) = 1,$$
$$v_f(x, y) = 1, \quad w_h(x, y) = 1.$$

The terms behind the minus sign correspond to the work of the internal forces. The other terms represent the work done by the forces external relative to the column.

Substitution of (6.4) and (6.5) in (6.6) yields the following system of partial differential equations for the functions $u_i(x, y)$, $v_g(x, y)$, $w_k(x, y)$:

$$\left.\begin{aligned}
&\sum_{i=1}^{m} a_{ji} \left(\frac{\partial^2 u_i}{\partial x^2} + \frac{1-\nu_0}{2} \frac{\partial^2 u_i}{\partial y^2} \right) - \frac{1-\nu_0}{2} \sum_{i=1}^{m} b_{ji} u_i + \\
&+ \frac{1+\nu_0}{2} \sum_{g=1}^{l} t_{jg} \frac{\partial^2 v_g}{\partial x \, \partial y} + \sum_{k=1}^{n} \left(\nu_0 d_{jk} - \frac{1-\nu_0}{2} c_{jk} \right) \frac{\partial w_k}{\partial x} + \frac{1-\nu_0^2}{E_0} p_j = 0 \\
&\qquad (j = 1, 2, 3, \ldots, m), \\
&\sum_{g=1}^{l} m_{fg} \left(\frac{\partial^2 v_g}{\partial y^2} + \frac{1-\nu_0}{2} \frac{\partial^2 v_g}{\partial x^2} \right) - \frac{1-\nu_0}{2} \sum_{g=1}^{l} n_{fg} v_g + \\
&+ \frac{1+\nu_0}{2} \sum_{i=1}^{m} t_{fi} \frac{\partial^2 u_i}{\partial x \, \partial y} + \sum_{k=1}^{n} \left(\nu_0 l_{fk} - \frac{1-\nu_0}{2} k_{fk} \right) \frac{\partial w_k}{\partial y} + \frac{1-\nu_0^2}{E_0} g_f = 0 \\
&\qquad (f = 1, 2, 3, \ldots, l), \\
&- \sum_{i=1}^{m} \left(\nu_0 d_{hi} - \frac{1-\nu_0}{2} c_{hi} \right) \frac{\partial u_i}{\partial x} - \sum_{g=1}^{l} \left(\nu_0 l_{hg} - \frac{1-\nu_0}{2} k_{hg} \right) \frac{\partial v_g}{\partial y} + \\
&\quad + \frac{1-\nu_0}{2} \sum_{k=1}^{n} r_{hk} \left(\frac{\partial^2 w_k}{\partial x^2} + \frac{\partial^2 w_k}{\partial y^2} \right) - \sum_{k=1}^{n} s_{hk} w_k + \frac{1-\nu_0^2}{E_0} q_h = 0 \\
&\qquad (h = 1, 2, 3, \ldots, n).
\end{aligned}\right\} \quad (6.7)$$

The coefficients in (6.7) are:

$$\left.\begin{array}{ll} a_{ji} = a_{ij} = \int \varphi_j \varphi_i \, dz, & m_{fg} = m_{gf} = \int \varkappa_f \varkappa_g \, dz, \\ b_{ji} = b_{ij} = \int \varphi_j' \varphi_i' \, dz, & n_{fg} = n_{gf} = \int \varkappa_f' \varkappa_g' \, dz, \\ c_{jk} = \int \varphi_j' \psi_k \, dz, & k_{fk} = \int \varkappa_f' \psi_k \, dz, \\ d_{jk} = \int \varphi_j \psi_k' \, dz, & l_{fk} = \int \varkappa_f \psi_k' \, dz, \\ r_{hk} = \int r_{kh} = \int \psi_h \psi_k \, dz, & c_{hi} = \int \psi_h \varphi_i' \, dz, \\ s_{hk} = s_{kh} = \int \psi_h' \psi_k' \, dz, & d_{hi} = \int \psi_h' \varphi_i \, dz, \\ k_{hg} = \int \psi_h \varkappa_g' \, dz, & t_{fi} = \int \varkappa_f \varphi_i \, dz, \\ l_{hg} = \int \psi_h' \varkappa_g \, dz, & t_{fg} = \int \varphi \, \varkappa_g \, dz. \end{array}\right\} \tag{6.8}$$

The definite integrals are taken over the entire height H of the elastic foundation.

The free terms in (6.7) represent the work done by the known external load over the corresponding virtual displacements:

$$\left.\begin{array}{l} p_i = \int p(x, y, z) \varphi_i(z) \, dz, \\ g_f = \int g(x, y, z) \varkappa_f(z) \, dz, \\ q_h = \int q(x, y, z) \psi_h(z) \, dz. \end{array}\right\} \tag{6.9}$$

When an external load acts on the elastic foundation, the integrals (6.9) are to be considered as Stieltjes integrals (cf. explanations to (1.12), (1.13)). If no body forces act, i.e., if the external load consists only of [distributed] surface forces $p(x, y), g(x, y), q(x, y)$, expressions (6.9) become:

$$\left.\begin{array}{l} p_i = p(x, y) \varphi_i(0), \\ g_f = g(x, y) \varkappa_f(0), \\ q_h = q(x, y) \psi_h(0), \end{array}\right\} \tag{6.10}$$

Differential equations (6.7) describe completely the states of strain and stress of an elastic foundation having a finite thickness H. The elastic foundation is considered to be an infinitely thick slab secured to its supporting surface and capable of sustaining normal and tangential loads.

The solution (6.7) for a thick isotropic plate is approximate from the viewpoint of the theory of elasticity. Its accuracy increases with the number of terms in (6.1). The differential equations (6.7) define at the same time a generalized three-dimensional model of the elastic foundation, whose properties depend on the number of terms in (6.1) and on the properties of the functions $\varphi_i(z)$, $\varkappa_g(z)$, $\psi_k(z)$. Different schemes, corresponding in varying degrees to the actual foundation, can be obtained by selecting different expressions for the functions φ_i, $\varkappa_g$, ψ_h.

The selection of these functions was discussed in detail in section 2, dealing with the plane strain of an elastic foundation. We repeat that this selection must be specific to the problem considered (e.g., according to an experimental law). In this case even the simplest model described by a minimum number of functions φ_i, $\varkappa_g$, ψ_k will be closer to reality than the model based on the postulation of the foundation modulus.

§ 7. THREE-DIMENSIONAL MODEL OF AN ELASTIC FOUNDATION WITH TWO CHARACTERISTICS

1

Consider an elastic foundation of finite thickness H (Figure 26). Let the horizontal displacements of the foundation vanish everywhere:

$$u(x, y, z) = 0, \quad v(x, y, z) = 0, \tag{7.1}$$

and let the vertical displacements be*:

$$w(x, y, z) = w(x, y)\psi(z), \tag{7.2}$$

where $\psi(z)$ is the function of the transverse distribution of the displacements, chosen in accordance with the nature of the problem.

By (7.1) and (7.2), only a single equation of (6.7) remains:

$$\frac{1-\nu_0}{2} r_{11}\nabla^2 w(x, y) - s_{11}w(x, y) + \frac{1-\nu_0^2}{E_0} q_1 = 0, \tag{7.3}$$

where

$$\nabla^2 w(x, y) = \frac{\partial^2 w(x, y)}{\partial x^2} + \frac{\partial^2 w(x, y)}{\partial y^2}, \tag{7.4}$$

$$r_{11} = \int_0^H \psi^2(z)\,dz, \qquad s_{11} = \int_0^H \psi'^2(z)\,dz. \tag{7.5}$$

The free term in (7.3) represents the work done by the known [external] load $q(x, y, z)$ over the virtual displacement $\bar{w}(x, y, z) = \psi(z)$ (for $w(x, y) = 1$) and is:

$$q_1 = \int_0^H q(x, y, z)\psi(z)\,dz. \tag{7.6}$$

If the external load consists only of surface forces $q(x, y)$, this becomes:

$$q_1 = q(x, y)\psi(0) \tag{7.7}$$

where $\psi(0)$ is the value of $\psi(z)$ at the surface of the elastic foundation.

Equation (7.3) can be rewritten as follows:

$$2t\nabla^2 w(x, y) - kw(x, y) + q_1 = 0. \tag{7.8}$$

* The subscript 1 in $w(x, y)$ and $\psi(z)$ will henceforth be omitted.

where

$$k=\frac{E_0 s_{11}}{1-\nu_0^2}, \qquad t=\frac{E_0 r_{11}}{4(1+\nu_0)}. \tag{7.9}$$

The coefficient k characterizes the compressive strain in the elastic foundation, and is thus analogous to the foundation modulus. The coefficient t characterizes the shearing strain in the elastic foundation.

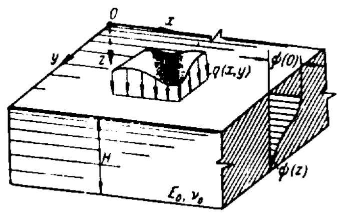


FIGURE 26.

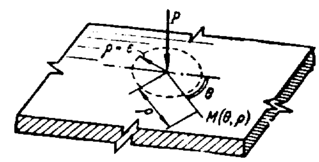


FIGURE 27.

The partial differential equation (7.8) differs from the relationship derived from the postulation of the foundation modulus by the term:

$$2t\nabla^2 w(x,\ y),$$

which makes allowance for the shearing stresses. In order to determine coefficients (7.9), we must specify $\psi(z)$. Assume that, in accordance with the problem, this function has the form:

$$\psi(z)=\frac{\operatorname{sh}\gamma(H-z)}{\operatorname{sh}\gamma H}, \tag{7.10}$$

where γ is a coefficient determining the variation with depth of the displacements.

Substitution of (7.10) in (7.9) yields [cf. (3.19)]:

$$\left.\begin{aligned} k&=\frac{E_0}{H(1-\nu_0^2)}\,\psi_k,\\ t&=\frac{E_0 H}{12(1+\nu_0)}\,\psi_t,\end{aligned}\right\} \tag{7.11}$$

where

$$\left.\begin{aligned} \psi_k&=\frac{\gamma H}{2}\left[\frac{\operatorname{sh}\gamma H\operatorname{ch}\gamma H+\gamma H}{\operatorname{sh}^2\gamma H}\right],\\ \psi_t&=\frac{3}{2}\,\frac{1}{\gamma H}\left[\frac{\operatorname{sh}\gamma H\operatorname{ch}\gamma H-\gamma H}{\operatorname{sh}^2\gamma H}\right].\end{aligned}\right\} \tag{7.12}$$

The elastic constants E_0 and ν_0 are (see (6.3)):

$$E_0=\frac{E_s}{1-\nu_s^2}, \qquad \nu_0=\frac{\nu_s}{1-\nu_s}. \tag{7.13}$$

We shall now calculate the displacements of an elastic foundation, due to a concentrated force P acting at the origin of the polar system of coordinates (θ, ρ) (Figure 27). [For $P \neq 0$], the following homogeneous differential equation is obtained:

$$\nabla^2 W - \alpha^2 W = 0, \tag{7.14}$$

where

$$\alpha = \sqrt{\frac{k}{2t}}. \tag{7.15}$$

Since the load is symmetrical with respect to the coordinate origin, the generalized vertical displacements will be independent of the angle θ. The Laplacian operator is in this case:

$$\nabla^2 W(\rho) = \frac{d^2 W(\rho)}{d\rho^2} + \frac{1}{\rho}\frac{dW(\rho)}{d\rho}. \tag{7.16}$$

By introducing a new variable

$$\xi = i\alpha\rho \tag{7.17}$$

we can reduce (7.14) to a Bessel equation of the imaginary argument ξ:

$$\frac{d^2 W}{d\xi^2} + \frac{1}{\xi}\frac{dW}{d\xi} + W = 0. \tag{7.18}$$

The general solution of (7.18) is*:

$$W = C_1 I_0(\alpha\rho) + C_2 K_0(\alpha\rho), \tag{7.19}$$

where $I_0(\alpha\rho)$ and $K_0(\alpha\rho)$ are modified zero-order Bessel functions of the first and second kind respectively, while C_1 and C_2 are arbitrary integration constants. Curves of $I_0(\alpha\rho)$ and $K_0(\alpha\rho)$, as functions of the argument $(\alpha\rho)$ are shown in Figure 28. It is seen that the behavior of these functions is similar to that of exponential functions.

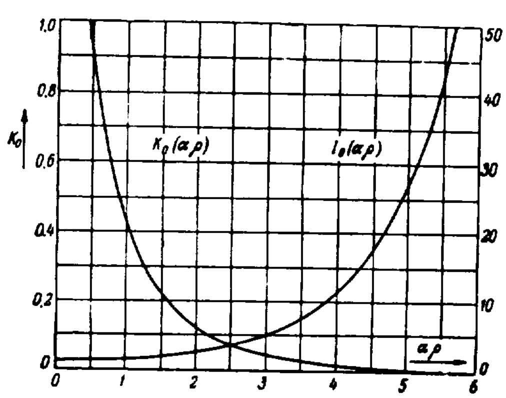


FIGURE 28.

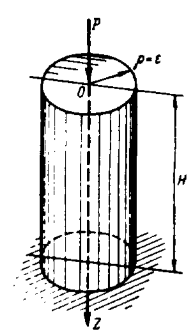


FIGURE 29.

* See, e.g., G.N. Watson. Theory of Bessel Functions.—Cambridge. 1923.

Since $I_0(\alpha\rho)$ and $K_0(\alpha\rho)$ are real for all values of ρ, the integration constants C_1 and C_2 will also be real. They can be found by considering the physical aspects of the problem. Since at infinity the displacements of the elastic foundation vanish, we obtain the following boundary condition:

at $\rho \to \infty$
$$W \to 0. \tag{7.20}$$

Since I_0 tends to infinity with ρ, it follows from (7.20) that:

$$C_1 = 0. \tag{7.21}$$

To determine C_2 from the equilibrium conditions, we cut an elementary cylinder of radius $\rho = \varepsilon$ ($\varepsilon \to 0$) from the elastic foundation (Figures 27 and 29). The equilibrium conditions for this cylinder can be written in the form:

$$\int_0^{2\pi} \rho\, d\theta \int_0^H \tau_{z\rho}\, \psi(z)\, dz + P\psi(0) = 0. \tag{7.22}$$

This equation represents the work done by the shearing stresses $\tau_{z\rho}$, distributed over the envelope of the cylinder, and by the external force P over the virtual displacement $\bar{w}(\rho, z) = \phi(z)$ (for $W(\rho) = 1$).

By analogy with (6.2), the shearing stresses $\tau_{z\rho}$ are expressed in the cylindrical system of coordinates (z, ρ) as follows:

$$\tau_{z\rho} = \frac{E_0}{2(1+\nu_0)} \frac{dW(\rho)}{d\rho} \phi(z). \tag{7.23}$$

Substitution of (7.23), (7.19), and (7.21) in (7.22) yields:

$$\int_0^{2\pi} 2\alpha t C_2 K_1(\alpha\rho)\, \rho\, d\theta = P\phi(0), \tag{7.24}$$

where, for $\alpha\rho \leqslant 1$

$$K_1(\alpha\rho) \approx \frac{1}{\alpha\rho},$$

[and t is given by (7.26)].

Integrating (7.24) we obtain for C_2:

$$C_2 = \frac{P\psi(0)}{4\pi t}, \tag{7.25}$$

where

$$t = \frac{E_0 r_{11}}{4(1+\nu_0)}, \quad r_{11} = \int_0^H \phi^2(z)\, dz. \tag{7.26}$$

From (7.19), (7.21), (7.25), and (7.2), we obtain the vertical displacement of any point of the elastic foundation:

$$w(\rho, z) = \frac{P\phi(0)}{4\pi t} K_0(\alpha\rho)\, \phi(z). \tag{7.27}$$

If, for instance, $\phi(z)$ is given by (7.10), expression (7.27) becomes:

$$w(\rho, z) = \frac{3P(1+\nu_0)}{\pi E_0 H \psi_t} K_0(\alpha\rho) \frac{\operatorname{sh}\gamma(H-z)}{\operatorname{sh}\gamma H}, \tag{7.28}$$

where [cf. (3.32)]

$$\left.\begin{aligned} \alpha &= \sqrt{\frac{k}{2t}} = \frac{1}{H}\cdot\frac{\sqrt{6(1-\nu_0)}}{1-\nu_0}\,\psi_\alpha, \\ \psi_t &= \frac{3}{2}\,\frac{1}{\gamma H}\left[\frac{\operatorname{sh}\gamma H\operatorname{ch}\gamma H-\gamma H}{\operatorname{sh}^2\gamma H}\right], \\ \psi_\alpha &= \gamma H\sqrt{\frac{1}{3}\,\frac{\operatorname{sh}\gamma H\operatorname{ch}\gamma H+\gamma H}{\operatorname{sh}\gamma H\operatorname{ch}\gamma H-\gamma H}}. \end{aligned}\right\} \tag{7.29}$$

If $\psi(z)$ is given by:

$$\psi(z) = e^{-\gamma z} \tag{7.30}$$

we obtain for an elastic semi-infinite space $(H \to \infty)$:

$$w(\rho,\ z) = \frac{2P\gamma(1+\nu_0)}{\pi E_0} K_0(\alpha\rho)\, e^{-\gamma z}, \tag{7.31}$$

where

$$\alpha = \sqrt{\frac{k}{2t}} = \gamma\,\frac{\sqrt{2(1-\nu_0)}}{1-\nu_0},$$

and γ is a coefficient of dimension 1/L which determines the variation of the displacements with depth.

As an example, Figure 30 shows displacements of the surface of the elastic foundation in units of $\frac{P}{\pi E_0}$, obtained from (7.31) for $\nu_0 = 0$. The Boussinesq curve corresponding to $\nu_0 = 0$ has also been plotted for comparison. It is seen that the cohesion of the elastic layer decreases with increasing γ, and the properties of the foundation approach those assumed by Winkler. On the other hand, when γ decreases the elastic foundation has a higher load-distributing capacity.

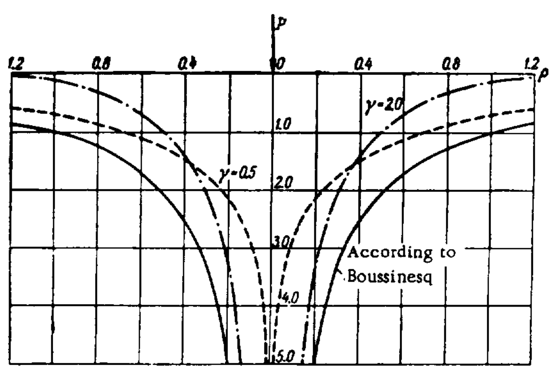


FIGURE 30.

Consider now an elastic foundation acted upon by a load, uniformly distributed within a circle of radius R (Figure 31).

Two regions exist in this case and, by (7.8), two differential equations:

$$\left.\begin{aligned} &\text{at } 0 \leqslant \rho \leqslant R \qquad \frac{d^2W_1}{d\rho^2} + \frac{1}{\rho}\frac{dW_1}{d\rho} - \alpha^2 W_1 = -\frac{q}{2t},\\ &\text{at } R \leqslant \rho < \infty \qquad \frac{d^2W_2}{d\rho^2} + \frac{1}{\rho}\frac{dW_2}{d\rho} - \alpha^2 W_2 = 0.\end{aligned}\right\} \tag{7.32}$$

The solutions of these equations are:

$$\left.\begin{aligned} W_1 &= C_1 I_0(\alpha\rho) + C_2 K_0(\alpha\rho) + \frac{q}{k},\\ W_2 &= C_3 I_0(\alpha\rho) + C_4 K_0(\alpha\rho)\end{aligned}\right\} \tag{7.33}$$

where k is given by (7.9).

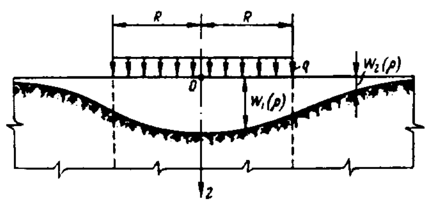


FIGURE 31.

The following boundary conditions are deduced from the nature of the problem for the determination of the integration constants C_1, C_2, C_3, C_4:

$$\left.\begin{aligned} &\text{at } \rho = 0 \qquad \frac{dW_1}{d\rho} = 0,\\ &\text{at } \rho \to \infty \qquad W_2 = 0,\end{aligned}\right\} \tag{7.34}$$

$$\left.\begin{aligned} &\text{at } \rho = R \qquad W_1 = W_2,\\ &\qquad\qquad \frac{dW_1}{d\rho} = \frac{dW_2}{d\rho}.\end{aligned}\right\} \tag{7.35}$$

It follows immediately from (7.34) that:

$$C_2 = C_3 = 0. \tag{7.36}$$

After substitution of (7.33), we can write (7.35) as follows:

$$\left.\begin{aligned} C_1 I_0(\alpha R) - C_4 K_0(\alpha R) &= -\frac{q}{k},\\ C_1 I_1(\alpha R) + C_4 K_1(\alpha R) &= 0,\end{aligned}\right\} \tag{7.37}$$

where I_1, K_1 are the first-order modified Bessel functions.

Solving the system (7.37) we obtain for C_1 and C_4:

$$\left.\begin{aligned} C_1 &= -\frac{q}{k}\,\frac{K_1(\alpha R)}{I_0(\alpha R)K_1(\alpha R) + I_1(\alpha R)K_0(\alpha R)},\\ C_4 &= \frac{q}{k}\,\frac{I_1(\alpha R)}{I_0(\alpha R)K_1(\alpha R) + I_1(\alpha R)K_0(\alpha R)}.\end{aligned}\right\} \tag{7.38}$$

Finally:

$$\left.\begin{aligned} W_1(\rho) &= C_1 I_0(\alpha\rho) + \frac{q}{k}, \\ W_2(\rho) &= C_4 K_0(\alpha\rho). \end{aligned}\right\} \qquad (7.39)$$

This problem could have been solved by proceeding from (7.27) which, for $z = 0$ and $P = 1$, defines the influence surface for the displacements. We then obtain for the displacement at $z = 0$:

$$W(0) = \frac{q\psi(0)}{4\pi t}\int_0^{2\pi}\int_0^R K_0(\alpha\rho)\,\rho\,d\rho\,d\theta = \frac{q\psi(0)}{k}\left[1 - \alpha R K_0(\alpha R)\right]. \qquad (7.40)$$

§ 8. THERMAL STRESSES IN AN ELASTIC FOUNDATION

1

In the design of foundations for heavy structures, it may be necessary to determine the stresses and strains caused by temperature variations. This problem is also encountered in the design of thick slabs and beams on rigid or elastic foundations.

Consider an elastic layer on a rigid foundation (Figure 5), and let this layer be in a state of plane strain as a result of a two-dimensional temperature field. In the general case the temperature is a function of the coordinates x, y and the time t:

$$T = T(x, y, t).$$

The problem will be solved by the variational method. The unknown displacements $u(x, y, t)$, $v(x, y, t)$ are expressed as finite series:

$$\left.\begin{aligned} u(x, y, t) &= \sum_{i=1}^{m} U_i(x, t)\,\varphi_i(y) \qquad (i = 1, 2, 3, \ldots, m), \\ v(x, y, t) &= \sum_{k=1}^{n} V_k(x, t)\,\psi_k(y) \qquad (k = 1, 2, 3, \ldots, n), \end{aligned}\right\} \qquad (8.1)$$

where the functions $U_i(x, t)$, $V_k(x, t)$ are the unknowns, while the functions $\varphi_i(y)$, $\psi_k(y)$ are chosen according to the nature of the problem.

The following system of $m + n$ equations is obtained as before ((1.7) and (1.8)) by considering the generalized equilibrium conditions of an elementary column of height H and measuring $1 \times \delta$ in plan, and assuming that no surface or body forces act on the elastic foundation:

$$\left.\begin{aligned} &\int \frac{\partial \sigma_x}{\partial x}\varphi_j\,dF - \int \tau_{xy}\varphi_j'\,dF = 0 \qquad (j = 1, 2, 3, \ldots, m), \\ &\int \frac{\partial \tau_{yx}}{\partial x}\psi_h\,dF - \int \sigma_y\psi_h'\,dF = 0 \qquad (h = 1, 2, 3, \ldots, n), \end{aligned}\right\} \qquad (8.2)$$

where $dF = \delta\, dy$

The strain components are:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x};$$

the stresses σ_x, σ_y, τ_{xy} are, in the case of plane strain caused by temperature variations:

$$\left.\begin{aligned}
\sigma_x &= \frac{E_0}{1-\nu_0^2}\left(\frac{\partial u}{\partial x} + \nu_0\frac{\partial v}{\partial y}\right) - \frac{\alpha E_0 T}{1-\nu_0},\\
\sigma_y &= \frac{E_0}{1-\nu_0^2}\left(\frac{\partial v}{\partial y} + \nu_0\frac{\partial u}{\partial x}\right) - \frac{\alpha E_0 T}{1-\nu_0},\\
\tau_{xy} &= \frac{E_0}{2(1+\nu_0)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),
\end{aligned}\right\} \tag{8.3}$$

where α is the coefficient of linear expansion; $T = T(x, y, t)$ is the temperature at point (x, y) and time t; E_0 and ν_0 are elastic constants defined by (2.1).

The following system of $m+n$ ordinary differential equations in the unknowns $U_i(x, t)$, $V_k(x, t)$ is obtained by first inserting (8.1) into (8.3), and then the resulting expressions into (8.2):

$$\left.\begin{aligned}
&\sum_{i=1}^{m} a_{ji} U_i'' - \frac{1-\nu_0}{2}\sum_{i=1}^{m} b_{ji} U_i + \sum_{k=1}^{n}\left(\nu_0 t_{jk} - \frac{1-\nu_0}{2} c_{jk}\right) V_k' - \\
&\qquad\qquad - \alpha(1+\nu_0) X_{jT} = 0\\
&\qquad (j = 1, 2, 3, \ldots, m),\\
&-\sum_{i=1}^{m}\left(\nu_0 t_{hi} - \frac{1-\nu_0}{2} c_{hi}\right) U_i' + \frac{1-\nu_0}{2}\sum_{k=1}^{n} r_{hk} V_k'' - \\
&\qquad\qquad - \sum_{k=1}^{n} s_{hk} V_k + \alpha(1+\nu_0) Y_{hT} = 0\\
&\qquad (h = 1, 2, 3, \ldots, n).
\end{aligned}\right\} \tag{8.4}$$

The coefficients a_{ji}, b_{ji}, $\ldots$, r_{hk}, s_{hk} in (8.4) are given as before by (1.11) as functions of $\varphi_i(y)$, $\psi_k(y)$. The free terms X_{jT} and Y_{hT} are:

$$X_{jT} = \int \frac{\partial T}{\partial x}\varphi_j\, dF, \quad Y_{hT} = \int T\psi_h'\, dF. \tag{8.5}$$

Equations (8.4), together with the corresponding boundary conditions, completely define the temperature equilibrium of a layer of finite thickness H in a state of plane strain. This method can also be applied to the design of elastic foundations and thick plates in a state of three-dimensional stress. The three-dimensional thermo-elastic problem can be reduced to a two-dimensional problem by the method used in section 6 for an elastic foundation subjected to an external load.

2

As an example, consider an elastic layer of finite length in the x direction, which is in a state of plane strain (Figure 32). Let the layer be rigidly connected to its base, so that the displacements $u(x, y)$ and $v(x, y)$ in the plane of contact between layer and base vanish.

Taking only the first three terms in (8.1), we obtain the following approximations:

$$\left.\begin{aligned} u(x, y, t) &= U_1(x, t)\varphi_1(y) + U_2(x, t)\varphi_2(y) + U_3(x, t)\varphi_3(y), \\ v(x, y, t) &= V_1(x, t)\psi_1(y) + V_2(x, t)\psi_2(y) + V_3(x, t)\psi_3(y). \end{aligned}\right\} \quad (8.6)$$

The functions $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_3(y)$, $\psi_1(y)$, $\psi_2(y)$, $\psi_3(y)$ are represented in Figure 33. From (1.11) and Figure 33, we obtain the coefficients in (8.4):

$$\left.\begin{aligned}
& a_{11} = r_{11} = \frac{\delta H}{9}, \\
& a_{12} = a_{21} = r_{12} = r_{21} = a_{23} = a_{32} = r_{23} = r_{32} = \frac{\delta H}{18}, \\
& a_{13} = a_{31} = r_{13} = r_{31} = 0, \\
& a_{22} = a_{33} = \frac{2\delta H}{9}, \\
& b_{11} = s_{11} = \frac{3\delta}{H}, \\
& b_{12} = b_{21} = s_{12} = s_{21} = b_{23} = b_{32} = s_{23} = s_{32} = -\frac{3\delta}{H}, \\
& b_{13} = b_{31} = s_{13} = s_{31} = 0, \\
& b_{22} = b_{33} = \frac{6\delta}{H}, \\
& c_{11} = c_{12} = t_{11} = t_{21} = c_{23} = t_{32} = -\frac{\delta}{2}, \\
& c_{13} = t_{13} = c_{22} = t_{22} = c_{31} = t_{31} = c_{33} = t_{33} = 0, \\
& c_{21} = t_{12} = c_{32} = t_{23} = \frac{\delta}{2}.
\end{aligned}\right\} \quad (8.7)$$

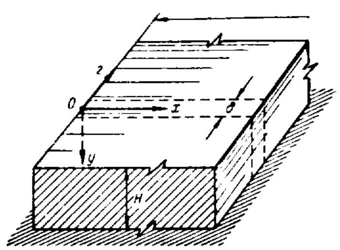


FIGURE 32.

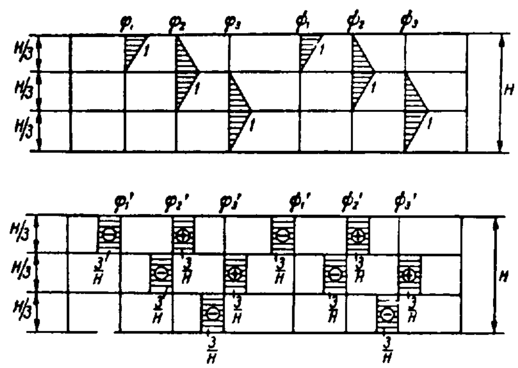


FIGURE 33.

By substituting these values in (8.4), a system of six differential equations in the six unknowns U_1, U_2, U_3, V_1, V_2, V_3 can be represented in the form of Table 1. In this table D and D^2 denote respectively the first and second - order differential operators on the function given at the head of the column. The terms A_1, A_2, A_3, B_1, B_2, B_3, in the last column on the right of Table 1 are:

$$\left.\begin{aligned}
A_1 &= \alpha(1+\nu_0)\int \frac{\partial T}{\partial x}\varphi_1\,dy, & B_1 &= -\alpha(1+\nu_0)\int T\psi_1'\,dy, \\
A_2 &= \alpha(1+\nu_0)\int \frac{\partial T}{\partial x}\varphi_2\,dy, & B_2 &= -\alpha(1+\nu_0)\int T\psi_2'\,dy, \\
A_3 &= \alpha(1+\nu_0)\int \frac{\partial T}{\partial x}\varphi_3\,dy, & B_3 &= -\alpha(1+\nu_0)\int T\psi_3'\,dy.
\end{aligned}\right\} \quad (8.8)$$

TABLE 1

U_1	U_2	U_3	V_1	V_2	V_3	
$\frac{H}{9}D^2 - \frac{3}{2}\frac{1-\nu_0}{H}$	$\frac{H}{18}D^2 + \frac{3}{2}\frac{1-\nu_0}{H}$	—	$\frac{1-3\nu_0}{4}D$	$\frac{1+\nu_0}{4}D$	—	A_1
$\frac{H}{18}D^2 + \frac{3}{2}\frac{1-\nu_0}{H}$	$2\left(\frac{H}{9}D^2 - \frac{3}{2}\frac{1-\nu_0}{H}\right)$	$\frac{H}{18}D^2 + \frac{3}{2}\frac{1-\nu_0}{H}$	$-\frac{1+\nu_0}{4}D$	—	$\frac{1+\nu_0}{4}D$	A_2
—	$\frac{H}{18}D^2 + \frac{3}{2}\frac{1-\nu_0}{H}$	$2\left(\frac{H}{9}D^2 - \frac{3}{2}\frac{1-\nu_0}{H}\right)$	—	$-\frac{1+\nu_0}{4}D$	—	A_3
$-\frac{1-3\nu_0}{4}D$	$\frac{1+\nu_0}{4}D$	—	$\frac{1-\nu_0}{18}HD^2 - \frac{3}{H}$	$\frac{1-\nu_0}{36}HD^2 + \frac{3}{H}$	—	B_1
$-\frac{1+\nu_0}{4}D$	—	$\frac{1+\nu_0}{4}D$	$\frac{1-\nu_0}{36}HD^2 + \frac{3}{H}$	$2\left(\frac{1-\nu_0}{18}HD^2 - \frac{3}{H}\right)$	$\frac{1-\nu_0}{36}HD^2 + \frac{3}{H}$	B_2
—	$-\frac{1+\nu_0}{4}D$	—	—	$\frac{1-\nu_0}{36}HD^2 + \frac{3}{H}$	$2\left(\frac{1-\nu_0}{18}HD^2 - \frac{3}{H}\right)$	B_3

When the function $T = T(x, y, t)$ is known, the differential equations in Table 1 can be integrated by usual methods. In accordance with the variational method described above, the boundary conditions at $x = 0$ and $x = l$ have to be given in generalized form. Thus, in the case of free ends they can be written in the form (cf. (1.14)):

$$\int \sigma_x \varphi_i \, dF = 0, \qquad \int \tau_{xy} \psi_h \, dF = 0. \tag{8.9}$$

When the ends are built-in, and both horizontal and vertical displacements are prevented in sections $x = 0$ and $x = l$, the generalized displacements must be zero:

$$U_i = 0, \quad V_k = 0. \tag{8.10}$$

We shall later discuss the case when diaphragms, rigid in their plane and flexible outside their plane, are located at $x = 0$ and $x = l$. Such diaphragms prevent only vertical displacements. The stresses σ_x in the end sections then vanish. These boundary conditions are written as follows in a generalized form:

$$\int \sigma_x \varphi_i \, dF = 0, \quad V_k = 0. \tag{8.11}$$

Let the function $T(x, y, t)$ be expressed as a trigonometric series:

$$T(x, y, t) = \sum_{n=1}^{\infty} T_n(y, t) \sin \frac{(2n-1)\pi x}{l}. \tag{8.12}$$

The Fourier coefficients are:

$$T_n(y, t) = \frac{2}{l} \int_0^l T(x, y, t) \sin \frac{(2n-1)\pi x}{l} dx. \tag{8.13}$$

It will be assumed that the boundary conditions of the problem are given by (8.11). In this case, the solution of the differential equations in Table 1 can be approximated by trigonometric series. Writing the unknown functions $U_i(x, t)$, $V_k(x, t)$ in the form:

$$\left.\begin{aligned} U_i(x, t) &= \sum_{n=1}^{\infty} U_{in}(t) \cos \frac{(2n-1)\pi x}{l}, \\ V_k(x, t) &= \sum_{n=1}^{\infty} V_{kn}(t) \sin \frac{(2n-1)\pi x}{l}. \end{aligned}\right\} \tag{8.14}$$

it is easily seen that they satisfy (8.11).

Substitution of (8.12) and (8.14) in Table 1 yields a system of algebraic equations for the determination of the coefficients $U_{in}(t)$, $V_{kn}(t)$ in (8.14). Six equations in six unknowns $U_{1n}, U_{2n}, U_{3n}, V_{1n}, V_{2n}, V_{3n}$ correspond to each value of n (Table 2). The free terms in these equations are:

$$\left.\begin{aligned} A_{in} &= (2n-1)\pi(1+\nu_0)\alpha \int_0^H T_n(y, t)\varphi_i\, dy \qquad (i = 1, 2, 3), \\ B_{kn} &= -(1+\nu_0)\alpha l \int_0^H T_n(y, t)\psi_k\, dy \qquad (k = 1, 2, 3). \end{aligned}\right\} \tag{8.15}$$

The solution of this problem has thus been reduced to solving a system of algebraic equations. This must be done in a generalized form, since the free terms A_{in}, B_{kn} are functions of t.

After $U_{1n}(t)$, $U_{2n}(t)$, $U_{3n}(t)$, $V_{1n}(t)$, $V_{2n}(t)$, $V_{3n}(t)$ have been determined, the displacements of the elastic layer can be obtained for any instant from (8.14) and (8.6). The stresses are found from (8.3), after insertion of (8.6) and (8.14). The following expressions are obtained for the normal and shearing stresses of the elastic layer:

$$\sigma_x = \frac{E_0}{1-\nu_0^2} \sum_{n=1}^{\infty} \left[-\frac{(2n-1)\pi}{l} \sum_{i=1}^{3} U_{in}(t)\varphi_i(y) + \right.$$
$$\left. + \nu_0 \sum_{k=1}^{3} V_{kn}(t)\psi_k(y) - \alpha(1+\nu_0) T_n(y, t) \right] \sin \frac{(2n-1)\pi x}{l}, \tag{8.16}$$

TABLE 2

No.	U_{1n}	U_{2n}	U_{3n}	V_{1n}	V_{2n}	V_{3n}	Free
1	$-\left[\frac{(2n-1)^2\pi^2}{9}\frac{H}{l}+ + \frac{3}{2}(1-\nu_0)\frac{l}{H}\right]$	$-\left[\frac{(2n-1)^2\pi^2}{18}\frac{H}{l}- - \frac{3}{2}(1-\nu_0)\frac{l}{H}\right]$		$\frac{\pi}{4}(2n-1)(1-3\nu_0)$	$\frac{\pi}{4}(2n-1)(1+\nu_0)$	—	A_{1n}
2	$-\left[\frac{(2n-1)^2\pi^2}{18}\frac{H}{l}- - \frac{3}{2}(1-\nu_0)\frac{l}{H}\right]$	$-2\left[\frac{(2n-1)^2\pi^2}{9}\frac{H}{l}+ + \frac{3}{2}(1-\nu_0)\frac{l}{H}\right]$	$-\left[\frac{(2n-1)^2\pi^2}{18}\frac{H}{l}- - \frac{3}{2}(1-\nu_0)\frac{l}{H}\right]$	$-\frac{\pi}{4}(2n-1)(1+\nu_0)$	—	$\frac{\pi}{4}(2n-1)(1+\nu_0)$	A_{2n}
3	—	$-\left[\frac{(2n-1)^2\pi^2}{18}\frac{H}{l}- - \frac{3}{2}(1-\nu_0)\frac{l}{H}\right]$	$-2\left[\frac{(2n-1)^2\pi^2}{9}\frac{H}{l}+ + \frac{3}{2}(1-\nu_0)\frac{l}{H}\right]$	—	$-\frac{\pi}{4}(2n-1)(1+\nu_0)$	—	A_{3n}
4	$\frac{\pi}{4}(2n-1)(1-3\nu_0)$	$-\frac{\pi}{4}(2n-1)(1+\nu_0)$	—	$-\left[\frac{\pi^2(2n-1)^2(1-\nu_0)}{18}\times \times\frac{H}{l}+3\frac{l}{H}\right]$	$-\left[\frac{\pi^2(2n-1)^2(1-\nu_0)}{36}\times \times\frac{H}{l}-3\frac{l}{H}\right]$	—	B_{1n}
5	$\frac{\pi}{4}(2n-1)(1+\nu_0)$	—	$-\frac{\pi}{4}(2n-1)(1+\nu_0)$	$-\left[\frac{\pi^2(2n-1)^2(1-\nu_0)}{36}\times \times\frac{H}{l}-3\frac{l}{H}\right]$	$-2\left[\frac{\pi^2(2n-1)^2(1-\nu_0)}{18}\times \times\frac{H}{l}+3\frac{l}{H}\right]$	$-\left[\frac{\pi^2(2n-1)^2(1-\nu_0)}{36}\times \times\frac{H}{l}-3\frac{l}{H}\right]$	B_{2n}
6	—	$\frac{\pi}{4}(2n-1)(1+\nu_0)$	—	—	$-\left[\frac{\pi^2(2n-1)^2(1-\nu_0)}{36}\times \times\frac{H}{l}-3\frac{l}{H}\right]$	$-2\left[\frac{\pi^2(2n-1)^2(1-\nu_0)}{18}\times \times\frac{H}{l}+3\frac{l}{H}\right]$	B_{3n}

$$\sigma_y = \frac{E_0}{1-\nu_0^2}\sum_{n=1}^{\infty}\left[\sum_{k=1}^{3} V_{kn}(t)\,\psi_k'(y) - \nu_0\frac{(2n-1)\pi}{l}\sum_{i=1}^{3} U_{in}(t)\,\varphi_i(y) - \right.$$

$$\left. - \alpha(1+\nu_0)\,T_n(y,\ t)\right]\sin\frac{(2n-1)\pi x}{l} \qquad \text{(8.16) cont'd}$$

$$\tau_{xy} = \frac{E_0}{2(1+\nu_0)}\sum_{n=1}^{\infty}\left[\sum_{i=1}^{3} U_{in}(t)\,\varphi_i'(y) + \right.$$

$$\left. + \frac{(2n-1)\pi}{l}\sum_{k=1}^{3} V_{kn}'(t)\,\psi_k(y)\right]\cos\frac{(2n-1)\pi x}{l}\ .$$

From (8.16), we can determine the stresses in any section of the elastic layer for the duration of temperature variation, provided $T(x,\ y,\ t)$ is known. It is thus possible to find the stresses in a block of concrete caused by the temperature variation during the setting and hardening of the concrete, or by its contraction, if the latter can be analytically expressed. *

* See, e. g., Cheche's papers /81, 82/.

Chapter II

BENDING OF A BEAM ON AN ELASTIC FOUNDATION

§ 1. DIFFERENTIAL EQUATION OF BENDING OF A BEAM ON AN ELASTIC FOUNDATION WITH TWO CHARACTERISTICS

Consider a beam lying on the surface of a single-layer elastic foundation. Let an external load $p(x)$ act on the beam (Figure 34). It will be assumed that the sections remain plane during bending, and that friction between the beam and the foundation can be neglected. The differential equation of bending of the beam is then:

$$EJV^{IV}(x) = p(x) - q(x), \tag{1.1}$$

where $q(x)$ is the reaction of the elastic foundation (= load acting on foundation), and $V(x)$ is the beam deflection [V^{IV} represents $\frac{d^4V}{dx^4}$].

Equation (1.1) contains two unknown functions $V(x)$ and $q(x)$. In order to determine them it is necessary to establish the relationship between the load acting on the foundation and the displacements. This relationship is obtained from the condition that the deflection of the beam is everywhere equal to the vertical displacement of the foundation.

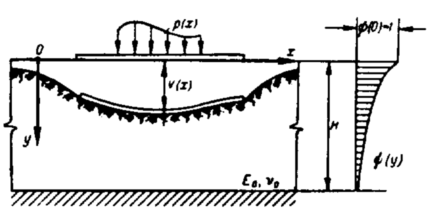


FIGURE 34.

The equation of equilibrium for a single-layer foundation is* [cf. (3.7), (3.8), (3.9) of Chapter I]:

$$-2tV'' + kV = q(x)\,\psi(0), \tag{1.2}$$

* The subscript 1 is henceforth omitted in $V(y)$ and $\psi(y)$.

where

$$
\left.\begin{aligned}
k &= \frac{E_0\delta}{1-\nu_0^2}\int_0^H \psi'^2(y)\,dy, \\
t &= \frac{E_0\delta}{4(1+\nu_0)}\int_0^H \psi^2(y)\,dy.
\end{aligned}\right\} \tag{1.3}
$$

It is most convenient to select the function $\psi(y)$ in such a way that $\psi(0)=1$. The generalized displacement $V(x)$ will then represent the displacement of the surface of the elastic foundation, and equation (1.2) becomes:

$$
-2tV'' + kV = q(x). \tag{1.4}
$$

Since the deflection of the beam equals the vertical displacement of the surface of the elastic foundation, equations (1.1) and (1.4) can be considered together:

$$
\left.\begin{aligned}
-2tV'' + kV &= q(x), \\
EJV^{IV} &= p(x) - q(x).
\end{aligned}\right\} \tag{1.5}
$$

Elimination of $q(x)$ from these two equations yields:

$$
EJV^{IV} - 2tV'' + kV = p(x). \tag{1.6}
$$

This equation differs from the equation, derived by postulating the foundation modulus, by the term containing the second derivative which makes allowance for the influence of shearing stresses in the elastic foundation.

We introduce the dimensionless coordinate $\eta = \frac{x}{L}$, where:

$$
L = \sqrt[3]{\frac{2EJ(1-\nu_0^2)}{E_0\delta}}, \tag{1.7}
$$

is the "elastic characteristic of the beam."*

Equation (1.6) then becomes:

$$
\frac{d^4V}{d\eta^4} - 2r^2\frac{d^2V}{d\eta^2} + s^4V = \frac{pL^4}{EJ}. \tag{1.8}
$$

where

$$
\left.\begin{aligned}
r^2 &= \frac{tL^2}{EJ} = \frac{1}{2}\frac{1-\nu_0}{L}\int_0^H \psi_1^2\,dy, \\
s^4 &= \frac{kL^4}{EJ} = 2L\int_0^H \psi_1'^2\,dy.
\end{aligned}\right\} \tag{1.9}
$$

Equation (1.8) represents the generalized equilibrium conditions of the elastic layer together with the beam lying on its surface. Hence, $V(\eta)$ is the generalized vertical displacement.

* A similar expression was adopted by Gorbunov-Posadov for analyzing a beam of infinite length /25, 26/.

To the generalized displacement $V(\eta)$ and slope $\varphi(\eta)=\frac{1}{L}V'(\eta)$ (in this case, these magnitudes represent the actual displacement and slope of the beam) there corresponds a generalized shearing force representing the shearing stresses. This force is distinct from the shearing force Q acting on the beam. In accordance with the variational method employed before (cf. second equation (1.14) of Chapter I) (see table on page 82):

$$N(\eta)=-\frac{EJ}{L^3}\left[V'''(\eta)-2r^2V'(\eta)\right]. \tag{1.10}$$

This expression must be taken into account for the boundary conditions which, as mentioned in Chapter I, have to be in integral form.

When $V(\eta)$ has been determined, the reactions $q(\eta)$ can be found from (1.4). The bending moments and shearing forces are:

$$M=-EJ\frac{d^2V}{dx^2}=-\frac{EJ}{L^2}\frac{d^2V}{d\eta^2}, \tag{1.11}$$

$$Q=-EJ\frac{d^3V}{dx^3}=-\frac{EJ}{L^3}\frac{d^3V}{d\eta^3}. \tag{1.12}$$

The solution obtained corresponds to the two-dimensional problem of the theory of elasticity. Hence, (1.6) (or (1.8)) is valid both for a beam lying on a vertical foundation of equal width δ (Figure 35), and for strips of width δ cut in the transverse direction from a long plate (Figure 36) lying on an elastic foundation. These two cases correspond respectively to plane stress and plane strain.

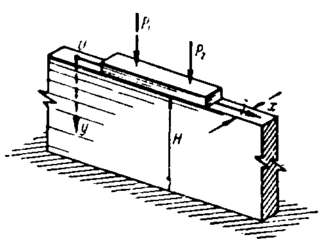


FIGURE 35.

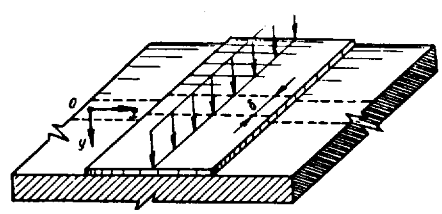


FIGURE 36.

In the case of plane strain, we have:

$$E_0=\frac{E_s}{1-\nu_s^2}, \qquad \nu_0=\frac{\nu_s}{1-\nu_s}, \tag{1.13}$$

where E_s, ν_s are respectively the modulus of elasticity and Poisson's ratio for the elastic foundation.

Furthermore, in (1.11) and (1.12), the equivalent moment of inertia J of the strip is:

$$J=\frac{\delta h^3}{12(1-\mu^2)},$$

where μ is Poisson's ratio for the material of the strip.

2. SOLUTION OF THE GENERALIZED EQUATION OF EQUILIBRIUM BY MEANS OF PARTICULAR INTEGRALS

1

To solve (1.8), we must first find the general integral of the corresponding homogeneous equation:

$$\frac{d^4V}{d\eta^4} - 2r^2\frac{d^2V}{d\eta^2} + s^4V = 0. \tag{2.1}$$

which is:

$$V(\eta) = C_1\Phi_1 + C_2\Phi_2 + C_3\Phi_3 + C_4\Phi_4, \tag{2.2}$$

where C_1, C_2, C_3, C_4 are integration constants and Φ_1, Φ_2, Φ_3, Φ_4, are roots of the auxiliary equation:

$$k^4 - 2r^2k^2 + s^4 = 0 \quad * \tag{2.3}$$

Since neither s nor r can be negative, the ratio $\frac{s}{r}$ is always positive. The solution of the auxiliary equation is:

1) for $s > r$

$$k = \pm\bar{\alpha} \pm \bar{\beta}i, \tag{2.4}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are real and positive:

$$\bar{\alpha} = \sqrt{\frac{s^2+r^2}{2}}, \quad \bar{\beta} = \sqrt{\frac{s^2-r^2}{2}}; \tag{2.5}$$

2) for $s = r$

$$\left.\begin{aligned} k_1 = k_2 = r, \\ k_3 = k_4 = -r; \end{aligned}\right\} \tag{2.6}$$

3) for $s < r$

$$\left.\begin{aligned} k_1 = -k_2 = \lambda_1 = \sqrt{r^2 + \sqrt{r^4 - s^4}}, \\ k_3 = -k_4 = \lambda_2 = \sqrt{r^2 - \sqrt{r^4 - s^4}}. \end{aligned}\right\} \tag{2.7}$$

The functions $\Phi_1, \Phi_2, \Phi_3, \Phi_4$, their first three derivatives, and their first integrals denoted by $\Phi_i^{(1)}$ corresponding respectively to (2.5), (2.6), and (2.7), are given in Table 3. The derivatives are expressed linearly through $\Phi_1, \Phi_2, \Phi_3, \Phi_4$.

The first case is the most important for the analysis of beams on single-layer foundations. When $s > r$, the functions in Table 3 differ from those for the bending of a beam on an elastic Winkler foundation by the arguments $\bar{\alpha}$ and $\bar{\beta}$ of the hyperbolic and trigonometric functions. These functions can be characterized by the ratio between the real and imaginary parts of the complex root (2.4):

$$\gamma = \frac{\bar{\beta}}{\bar{\alpha}} = \sqrt{\frac{s^2-r^2}{s^2+r^2}}, \tag{2.8}$$

* [k in this equation should not be confused with the characteristic k introduced in (3.9) of Chapter I.]

TABLE 3

Relative magnitude of s and r	Function and its derivatives	Φ_1 (odd)	Φ_2 (even)	Φ_3 (odd)	Φ_4 (even)
$s > r$	Φ	$\operatorname{sh} \bar{\alpha}\eta \cos \bar{\beta}\eta$	$\operatorname{ch} \bar{\alpha}\eta \cos \bar{\beta}\eta$	$\operatorname{ch} \bar{\alpha}\eta \sin \bar{\beta}\eta$	$\operatorname{sh} \bar{\alpha}\eta \sin \bar{\beta}\eta$
	Φ'	$\bar{\alpha}\Phi_2 - \bar{\beta}\Phi_4$	$\bar{\alpha}\Phi_1 - \bar{\beta}\Phi_3$	$\bar{\alpha}\Phi_4 + \bar{\beta}\Phi_2$	$\bar{\alpha}\Phi_3 + \bar{\beta}\Phi_1$
	Φ''	$(\bar{\alpha}^2 - \bar{\beta}^2)\,\Phi_1 - 2\bar{\alpha}\bar{\beta}\Phi_3$	$(\bar{\alpha}^2 - \bar{\beta}^2)\,\Phi_2 - 2\bar{\alpha}\bar{\beta}\Phi_4$	$(\bar{\alpha}^2 - \bar{\beta}^2)\,\Phi_3 + 2\bar{\alpha}\bar{\beta}\Phi_1$	$(\bar{\alpha}^2 - \bar{\beta}^2)\,\Phi_4 + 2\bar{\alpha}\bar{\beta}\Phi_2$
	Φ'''	$\bar{\alpha}\,(\bar{\alpha}^2 - 3\bar{\beta}^2)\,\Phi_2 +$ $+ \bar{\beta}\,(\bar{\beta}^2 - 3\bar{\alpha}^2)\,\Phi_4$	$\bar{\alpha}\,(\bar{\alpha}^2 - 3\bar{\beta}^2)\,\Phi_1 +$ $+ \bar{\beta}\,(\bar{\beta}^2 - 3\bar{\alpha}^2)\,\Phi_3$	$\bar{\alpha}\,(\bar{\alpha}^2 - 3\bar{\beta}^2)\,\Phi_4 -$ $- \bar{\beta}\,(\bar{\beta}^2 - 3\bar{\alpha}^2)\,\Phi_2$	$\bar{\alpha}\,(\bar{\alpha}^2 - 3\bar{\beta}^2)\,\Phi_3 -$ $- \bar{\beta}\,(\bar{\beta}^2 - 3\bar{\alpha}^2)\,\Phi_1$
$s = r$	Φ	$\operatorname{sh} r\eta$	$\operatorname{ch} r\eta$	$\eta \operatorname{ch} r\eta$	$\eta \operatorname{sh} r\eta$
	Φ'	$r\Phi_2$	$r\Phi_1$	$\Phi_2 + r\Phi_4$	$\Phi_1 + r\Phi_3$
	Φ''	$r^2\Phi_1$	$r^2\Phi_2$	$2r\Phi_1 + r^2\Phi_3$	$2r\Phi_2 + r^2\Phi_4$
	Φ'''	$r^3\Phi_2$	$r^2\Phi_1$	$3r^2\Phi_3 + r^3\Phi_4$	$3r^2\Phi_1 + r^3\Phi_3$
$s < r$	Φ	$\operatorname{sh} \lambda_1\eta$	$\operatorname{ch} \lambda_1\eta$	$\operatorname{sh} \lambda_2\eta$	$\operatorname{ch} \lambda_2\eta$
	Φ'	$\lambda_1\Phi_2$	$\lambda_1\Phi_1$	$\lambda_2\Phi_4$	$\lambda_2\Phi_3$
	Φ''	$\lambda_1^2\Phi_1$	$\lambda_1^2\Phi_2$	$\lambda_2^2\Phi_3$	$\lambda_2^2\Phi_4$
	Φ'''	$\lambda_1^3\Phi_2$	$\lambda_1^3\Phi_1$	$\lambda_2^3\Phi_4$	$\lambda_2^3\Phi_3$
$s = 0$	Φ	η	1	$\operatorname{sh} r_1\eta$	$\operatorname{ch} r_1\eta$
	Φ'	$\Phi_2 = 1$	0	$r_1\Phi_4$	$r_1\Phi_3$
	Φ''	0	0	$r_1^2\Phi_3$	$r_1^2\Phi_4$
	Φ'''	0	0	$r_1^3\Phi_4$	$r_1^3\Phi_3$
$s > r$	$\Phi^{(1)}$	$\dfrac{\bar{\alpha}\Phi_2 + \bar{\beta}\Phi_4}{\bar{\alpha}^2 + \bar{\beta}^2}$	$\dfrac{\bar{\alpha}\Phi_1 + \bar{\beta}\Phi_3}{\bar{\alpha}^2 + \bar{\beta}^2}$	$\dfrac{\bar{\alpha}\Phi_4 - \bar{\beta}\Phi_2}{\bar{\alpha}^2 + \bar{\beta}^2}$	$\dfrac{\bar{\alpha}\Phi_3 - \bar{\beta}\Phi_1}{\bar{\alpha}^2 + \bar{\beta}^2}$
$s = r$	$\Phi^{(1)}$	$\dfrac{1}{r}\,\Phi_2$	$\dfrac{1}{r}\,\Phi_1$	$\dfrac{1}{r}\,\Phi_4 - \dfrac{1}{r^2}\,\Phi_2$	$\dfrac{1}{r}\,\Phi_3 - \dfrac{1}{r^2}\,\Phi_1$
$s < r$	$\Phi^{(1)}$	$\dfrac{1}{\lambda_1}\,\Phi_2$	$\dfrac{1}{\lambda_1}\,\Phi_1$	$\dfrac{1}{\lambda_2}\,\Phi_4$	$\dfrac{1}{\lambda_2}\,\Phi_3$
$s = 0$	$\Phi^{(1)}$	$\dfrac{1}{2}\,\Phi_1^2 = \dfrac{\eta^2}{2}$	Φ_1	$\dfrac{1}{r_1}\,\Phi_4$	$\dfrac{1}{r_1}\,\Phi_3$

This ratio varies between 0 and 1. For $\gamma = 0$ the functions Φ (i = 1...4) degenerate into hyperbolic functions, multiplied by 1 or η. For $\gamma = 1$ they reduce to the functions of the bending of a beam on an elastic Winkler foundation.

The functions Φ_1, Φ_2, Φ_3, Φ_4, are tabulated in the appendix (Tables 1, 2, 3, 4) for values of γ between 0.0 and 1.0. In these tables $z = \alpha\eta$.

2

The general solution of the nonhomogeneous equation (1.8) is equal to the sum of the general homogeneous solution (2.2) and of a particular solution V_0:

$$V(\eta) = C_1\Phi_1 + C_2\Phi_2 + C_3\Phi_3 + C_4\Phi_4 + V_0. \tag{2.9}$$

When the distributed load is either constant or varies according to a linear law $p(\eta) = a + b\eta$, we can write:

$$V_0 = \frac{pL^4}{EJs^4}. \tag{2.10}$$

Although this method of determining $V(\eta)$ is very simple in principle, it involves cumbersome calculations. Even when the function $p(\eta)$ is defined by a single analytical expression for the entire beam, a system of four algebraic equations has to be solved in order to determine the four integration constants C_1, C_2, C_3, C_4. If the load varies according to different laws in different zones of the beam or includes concentrated forces, the general integral (2.9) will contain different integration constants in each zone, their total number being four times that of the zones. In the relatively simple problems represented in Figure 37, we have to find 12 constants by setting up 12 algebraic equations. Hence, the method described is only practical when the external load is given by a single analytical expression valid for the entire beam.

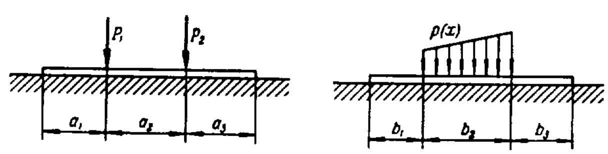


FIGURE 37.

To solve the problem considered in a practical and general manner, we shall apply Puzyrevskii-Krylov's method of the initial parameters (this method was first suggested by Cauchy) in the form proposed by Vlasov. This method requires the determination of only two integration constants irrespective of the load distribution*

* The method of initial parameters as applied to the analysis of beams (both ordinary and on elastic Winkler foundations) is explained in /73/ and in /47/.

§ 3. SOLUTION BY THE METHOD OF INITIAL PARAMETERS

1. General integral of the homogeneous equation

It will be assumed that the load consists of concentrated forces and moments. Between their points of application the bending of the beam is determined by the homogeneous equation (2.1) whose general integral (2.2) is written:

$$V = C_1K_1 + C_2K_2 + C_3K_3 + C_4K_4, \tag{3.1}$$

where K_1, K_2, K_3, K_4 are independent linear functions of $\Phi_1, \Phi_2, \Phi_3, \Phi_4$. The system K_1, K_2, K_3, K_4 can accordingly be called the fundamental system.

We choose the fundamental system in such a way that when $\eta = 0$, the generalized geometrical and statical magnitudes V, φ, M, N, expressed through the functions K_1, K_2, K_3, K_4 and their derivatives, form a unit matrix:

i	$V(0) = K_i(0)$	$\varphi(0) = \frac{1}{L}K_i'(0)$	$M(0) = -\frac{EJ}{L^2}K_i''(0)$	$N(0) = -\frac{EJ}{L^3}[K_i'''(0) - 2r^2K_i'(0)]$
1	1	0	0	0
2	0	1	0	0
3	0	0	1	0
4	0	0	0	1

Hence:

$$C_1 = V_0, \quad C_2 = \varphi_0, \quad C_3 = M_0, \quad C_4 = N_0,$$

[the subscript "O" denotes the values for $\eta = 0$].

The functions K_1, K_2, K_3, K_4 thus express the influence of unit initial parameters on the deflection, i.e., they are influence coefficients. Writing:

$$K_1 = K_{VV}, \quad K_2 = K_{V\varphi}, \quad K_3 = K_{VM}, \quad K_4 = K_{VN},$$

we obtain:

$$V(\eta) = V_0K_{VV} + \varphi_0K_{V\varphi} + M_0K_{VM} + N_0K_{VN}. \tag{3.2}$$

Inserting (3.2) into (1.10) and (1.11), the following system is obtained, including (3.2):

$$\left.\begin{aligned} V(\eta) &= V_0K_{VV} + \varphi_0K_{V\varphi} + M_0K_{VM} + N_0K_{VN}, \\ \varphi(\eta) &= V_0K_{\varphi V} + \varphi_0K_{\varphi\varphi} + M_0K_{\varphi M} + N_0K_{\varphi N}, \\ M(\eta) &= V_0K_{MV} + \varphi_0K_{M\varphi} + M_0K_{MM} + N_0K_{MN}, \\ N(\eta) &= V_0K_{NV} + \varphi_0K_{N\varphi} + M_0K_{NM} + N_0K_{NN}. \end{aligned}\right\} \tag{3.3}$$

The 16 influence functions K_{VV}, $K_{V\varphi}$, ..., K_{NN}, in (3.3) form a matrix for the direct linear transformation of V_0, φ_0, M_0, N_0 into V_η, φ_η, M_η, N_η. After the initial parameters have been determined, the problem can be considered as solved.

Any section $\eta = t$ in which V_t, φ_t, M_t, N_t are known can be taken as initial section. The values of V_η, φ_η, M_η, N_η in a section situated at a distance $\eta - t$ from the initial section will be determined by the same influence functions, provided the homogeneous differential equation of bending (2.1) is valid between these two sections:

$$\left.\begin{aligned} V_\eta &= V_t K_{VV} + \varphi_t K_{V\varphi} + M_t K_{VM} + N_t K_{VN},\\ \varphi_\eta &= V_t K_{\varphi V} + \varphi_t K_{\varphi\varphi} + M_t M_{\varphi M} + N_t K_{\varphi N},\\ M_\eta &= V_t K_{MV} + \varphi_t K_{M\varphi} + M_t K_{MM} + N_t K_{MN},\\ N_\eta &= V_t K_{NV} + \varphi_t K_{N\varphi} + M_t K_{NM} + N_t K_{NN}. \end{aligned}\right\} \qquad (3.4)$$

The influence functions K_{VV},..., K_{NN} are here functions of the argument $(\eta - t)$, while Φ_1,..., Φ_4 become $\Phi_1(\eta - t)$,..., $\Phi_4(\eta - t)$.

A very important property of matrices (3.4) and (3.3) is their symmetrical structure, as a result of which there are only 10 distinct influence functions. The four functions forming the secondary diagonal are not repeated. The remaining 12 functions, arranged symmetrically to this diagonal, are equal by pairs:

$$\left.\begin{aligned} K_{MM} &= K_{\varphi\varphi}, & K_{\varphi N} &= K_{VM}, & K_{MN} &= K_{V\varphi},\\ K_{NN} &= K_{VV}, & K_{N\varphi} &= K_{MV}, & K_{NM} &= K_{\varphi V}. \end{aligned}\right\} \qquad (3.5)$$

This property is derived from the reciprocity theorem of Maxwell and Betti. *

2. Effect of external load. General integral of the nonhomogeneous equation

Consider a beam of length l, acted upon by concentrated forces P_1, P_2,..., P_n (Figure 38) at points whose dimensionless coordinates are respectively t_1, t_2,..., t_n.

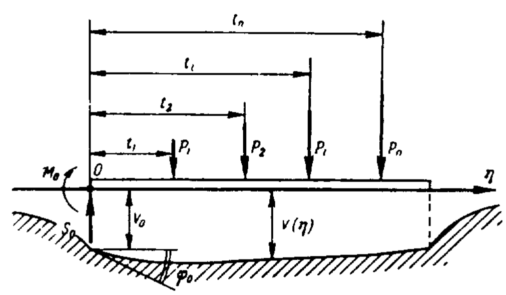


FIGURE 38.

* For a detailed discussion of influence functions forming matrices of direct and inverse transformation, cf. V. Z. Vlasov, "Tonkostennye uprugie sterzhni" (Thin-walled Elastic Bars).—Gostekhizdat. 1940. [Translated by IPST, No. 428.], and "Stroitel'naya mekhanika tonkostennykh prostranstvennykh sistem" (Structural Mechanics of Thin-walled Three-dimensional Systems).—Stroiizdat. 1949.

For $0 < \eta < t_1$ the kinematic and statical factors are determined by the initial parameters and influence functions:

$$\left.\begin{aligned}
V(\eta) &= V_0K_{VV}(\eta) + \varphi_0K_{V\varphi}(\eta) + M_0K_{VM}(\eta) + N_0K_{VN}(\eta),\\
\varphi(\eta) &= V_0K_{\varphi V}(\eta) + \varphi_0K_{\varphi\varphi}(\eta) + M_0K_{\varphi M}(\eta) + N_0K_{\varphi N}(\eta),\\
M(\eta) &= V_0K_{MV}(\eta) + \varphi_0K_{M\varphi}(\eta) + M_0K_{MM}(\eta) + N_0K_{MN}(\eta),\\
N(\eta) &= V_0K_{NV}(\eta) + \varphi_0K_{N\varphi}(\eta) + M_0K_{NM}(\eta) + N_0K_{NN}(\eta).
\end{aligned}\right\} \tag{3.6}$$

These expressions remain valid as long as the homogeneous differential equation (2.1) holds true, i.e., as long as $V(\eta)$, $\varphi(\eta)$, $M(\eta)$, $N(\eta)$ are continuous.

If one of these functions has a discontinuity at $\eta = t_k$, i.e., if a concentrated load acts at this point, the influence of this load must be taken into account for $\eta > t_k$, in accordance with the principle of superposition following from the linearity of (3.6). This influence is equal to the magnitude of the discontinuity multiplied by the corresponding influence function, calculated for the coordinate $(\eta - t_k)$.

Thus, $V(\eta)$, $\varphi(\eta)$, $M(\eta)$ remain continuous at all points where a concentrated force P_i acts, only $N(\eta)$ increasing by $(-P_i)$. Hence, for $t_1 < \eta < t_2$ (Figure 38):

$$\left.\begin{aligned}
V(\eta) &= V_0K_{VV}(\eta) + \varphi_0K_{V\varphi}(\eta) + M_0K_{VM}(\eta) + N_0K_{VN}(\eta) -\\
&\qquad - P_1K_{VN}(\eta - t_1),\\
\varphi(\eta) &= V_0K_{\varphi V}(\eta) + \varphi_0K_{\varphi\varphi}(\eta) + M_0K_{\varphi M}(\eta) + N_0K_{\varphi N}(\eta) -\\
&\qquad - P_1K_{\varphi N}(\eta - t_1),\\
M(\eta) &= V_0K_{MV}(\eta) + \varphi_0K_{M\varphi}(\eta) + M_0K_{MM}(\eta) + N_0K_{MN}(\eta) -\\
&\qquad - P_1K_{MN}(\eta - t_1),\\
N(\eta) &= V_0K_{NV}(\eta) + \varphi_0K_{N\varphi}(\eta) + M_0K_{NM}(\eta) + N_0K_{NN}(\eta) -\\
&\qquad - P_1K_{NN}(\eta - t_1).
\end{aligned}\right\} \tag{3.7}$$

For $t_i < \eta < t_{i+1}$ we have:

$$\left.\begin{aligned}
V(\eta) &= V_0K_{VV}(\eta) + \varphi_0K_{V\varphi}(\eta) + M_0K_{VM}(\eta) + N_0K_{VN}(\eta) -\\
&\qquad - \sum_{k=1}^{i} P_kK_{VN}(\eta - t_k),\\
\varphi(\eta) &= V_0K_{\varphi V}(\eta) + \varphi_0K_{\varphi\varphi}(\eta) + M_0K_{\varphi M}(\eta) + N_0K_{\varphi N}(\eta) -\\
&\qquad - \sum_{k=1}^{i} P_kK_{\varphi N}(\eta - t_k),\\
M(\eta) &= V_0K_{MV}(\eta) + \varphi_0K_{M\varphi}(\eta) + M_0K_{MM}(\eta) + N_0K_{MN}(\eta) -\\
&\qquad - \sum_{k=1}^{i} P_kK_{MN}(\eta - t_k),\\
N(\eta) &= V_0K_{NV}(\eta) + \varphi_0K_{N\varphi}(\eta) + M_0K_{NM}(\eta) + N_0K_{NN}(\eta) -\\
&\qquad - \sum_{k=1}^{i} P_kK_{NN}(\eta - t_k).
\end{aligned}\right\} \tag{3.8}$$

A distributed load $p(t)$ (Figure 39) can be considered as a system of elementary concentrated forces; we obtain for the loaded part of the beam:

$$\left.\begin{aligned}
V(\eta) &= V_0K_{VV}(\eta) + \varphi_0K_{V\varphi}(\eta) + M_0K_{VM}(\eta) + \\
&\quad + N_0K_{VN}(\eta) - \int_a^\eta p(t)\,K_{VN}(\eta - t)\,dt, \\
\varphi(\eta) &= V_0K_{\varphi V}(\eta) + \varphi_0K_{\varphi\varphi}(\eta) + M_0K_{\varphi M}(\eta) + \\
&\quad + N_0K_{\varphi N}(\eta) - \int_a^\eta p(t)\,K_{\varphi N}(\eta - t)\,dt, \\
M(\eta) &= V_0K_{MV}(\eta) + \varphi_0K_{M\varphi}(\eta) + M_0K_{MM}(\eta) + \\
&\quad + N_0K_{MN}(\eta) - \int_a^\eta p(t)\,K_{MN}(\eta - t)\,dt, \\
N(\eta) &= V_0K_{NV}(\eta) + \varphi_0K_{N\varphi}(\eta) + M_0K_{NM}(\eta) + \\
&\quad + N_0K_{NN}(\eta) - \int_a^\eta p(t)\,K_{NN}(\eta - t)\,dt.
\end{aligned}\right\} \qquad (3.9)$$

When distributed and concentrated loads act simultaneously, the integrals in (3.9) are Stieltjes integrals, i.e., the terms under the summation signs in (3.8) have to be added to them.

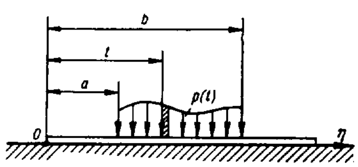


FIGURE 39.

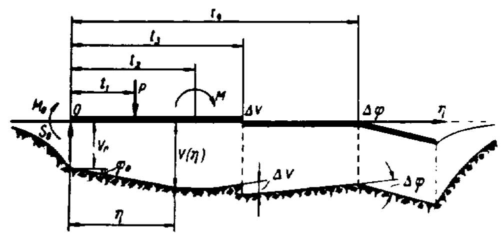


FIGURE 40.

In the most general case of arbitrary external influences, the solution of the nonhomogeneous differential equation (1.8) can be represented in the form:

$$\left.\begin{aligned}
V(\eta) &= V_0K_{VV} + \varphi_0K_{V\varphi} + M_0K_{VM} + N_0K_{VN} - F_V, \\
\varphi(\eta) &= V_0K_{\varphi V} + \varphi_0K_{\varphi\varphi} + M_0K_{\varphi M} + N_0K_{\varphi N} - F_\varphi, \\
M(\eta) &= V_0K_{MV} + \varphi_0K_{M\varphi} + M_0K_{MM} + N_0K_{MN} - F_M, \\
N(\eta) &= V_0K_{NV} + \varphi_0K_{N\varphi} + M_0K_{NM} + N_0K_{NN} - F_N,
\end{aligned}\right\} \qquad (3.10)$$

where F_V, F_φ, F_M, F_N are known functions depending on the load and its distribution. These "loads" need not be vertical forces and moments; they may also be breaks and abrupt bends in the beam.

For the case represented in Figure 40, we obtain for F_V and F_M:

$$\left.\begin{array}{l}
\text{at } 0<\eta<t_1 \quad F_V=F_M=0;\\
\text{at } t_1<\eta<t_2 \quad F_V=PK_{VN}(\eta-t_1),\\
\qquad F_M=PK_{MN}(\eta-t_1);\\
\text{at } t_2<\eta<t_3 \quad F_V=PK_{VN}(\eta-t_1)-MK_{VM}(\eta-t_2),\\
\qquad F_M=PK_{MN}(\eta-t_1)-MK_{MM}(\eta-t_2);\\
\text{at } t_3<\eta<t_4\\
F_V=PK_{VN}(\eta-t_1)-MK_{VM}(\eta-t_2)-\Delta VK_{VV}(\eta-t_3),\\
F_M=PK_{MN}(\eta-t_1)-MK_{MM}(\eta-t_2)-\Delta VK_{MV}(\eta-t_3);\\
\text{at } t_4<\eta\\
F_V=PK_{VN}(\eta-t_1)-MK_{VM}(\eta-t_2)-\Delta VK_{VV}(\eta-t_3)-\\
\qquad -\Delta\varphi K_{V\varphi}(\eta-t_4),\\
F_M=PK_{MN}(\eta-t_1)-MK_{MM}(\eta-t_2)-\Delta VK_{MV}(\eta-t_3)-\\
\qquad -\Delta\varphi K_{M\varphi}(\eta-t_4).
\end{array}\right\} \tag{3.11}$$

The initial parameters V_0, φ_0, M_0, N_0 can be obtained very simply by this method, the initial section of the beam being chosen arbitrarily. Thus, by selecting one of the beam ends as initial section $(\eta=0)$, we automatically determine two of the four parameters. The other two initial parameters can always be found from two equations defining the boundary conditions at the other end of the beam.

Thus, for a simply supported beam, we obtain respectively for $\eta=0$ and $\eta=\frac{l}{L}$:

$$V_0=0, \qquad M_0=0; \tag{3.12}$$

$$V\left(\frac{l}{L}\right)=0, \qquad M\left(\frac{l}{L}\right)=0. \tag{3.13}$$

Substitution of (3.10) and (3.12) in (3.13) yields:

$$V\left(\frac{l}{L}\right)=\varphi_0K_{V\varphi}+N_0K_{VN}-F_V=0,$$
$$M\left(\frac{l}{L}\right)=\varphi_0K_{M\varphi}+N_0K_{MN}-F_M=0,$$

where

$$K_{V\varphi},\ K_{VN},\ K_{M\varphi},\ K_{MN},\ F_V,\ F_M$$

are determined for

$$\eta=\frac{l}{L}\,(x=l)$$

3. Determination of the influence functions

The functions K_{VV}, $K_{V\varphi}, \ldots, K_{NN}$ in (3.10) were assumed to be known. It will be shown now how these functions are determined for the most important case $s > r$.

We proceed from the homogeneous differential equation (2.1) whose general integral, determining the generalized deflection $V(\eta)$, is

$$V(\eta) = C_1\Phi_1 + C_2\Phi_2 + C_3\Phi_3 + C_4\Phi_4. \tag{3.14}$$

The other kinematic and statical factors are linear functions of the derivatives of $V(\eta)$:

$$\left.\begin{aligned} \varphi(\eta) &= \frac{1}{L} V', \\ M(\eta) &= -\frac{EJ}{L^2} V'', \\ N(\eta) &= -\frac{EJ}{L^3} [V''' - 2r^2 V']. \end{aligned}\right\} \tag{3.15}$$

Substitution in (3.15) of (3.14) and the initial conditions (for $(\eta = 0)$ [cf. Table 3]

$$\Phi_2 = 1, \quad \Phi_1 = \Phi_3 = \Phi_4 = 0,$$
$$V = V_0, \quad \varphi = \varphi_0, \quad M = M_0, \quad N = N_0,$$

TABLE 4.

Influence functions

	V_0	φ_0	M_0	N_0
V_η	$K_{VV} = \Phi_2 - \frac{r^2}{2\bar{\alpha}\bar{\beta}} \Phi_4$	$K_{V\varphi} = \frac{L}{2\bar{\alpha}\bar{\beta}} (\bar{\beta}\Phi_1 + \bar{\alpha}\Phi_3)$	$K_{VM} = -\frac{L^2}{2\bar{\alpha}\bar{\beta}EJ} \Phi_4$	$K_{VN} = \frac{L^3}{2\bar{\alpha}\bar{\beta}s^2EJ} (\bar{\beta}\Phi_1 - \bar{\alpha}\Phi_3)$
φ_η	$K_{\varphi V} = \frac{s^2}{2\bar{\alpha}\bar{\beta}L} (\bar{\beta}\Phi_1 - \bar{\alpha}\Phi_3)$	$K_{\varphi\varphi} = \Phi_2 + \frac{r^2}{2\bar{\alpha}\bar{\beta}} \Phi_4$	$K_{\varphi M} = -\frac{L}{2\bar{\alpha}\bar{\beta}EJ} (\bar{\alpha}\Phi_3 + \bar{\beta}\Phi_1)$	$K_{\varphi N} = K_{VM}$
M_η	$K_{MV} = \frac{EJ}{2\bar{\alpha}\bar{\beta}L^2} s^4\Phi_4$	$K_{M\varphi} = -\frac{EJ}{2\bar{\alpha}\bar{\beta}L} [(3\bar{\alpha}^2 - \bar{\beta}^2)\bar{\beta}\Phi_1 - (\bar{\alpha}^2 - 3\bar{\beta}^2)\bar{\alpha}\Phi_3]$	$K_{MM} = K_{\varphi\varphi}$	$K_{MN} = K_{V\varphi}$
N_η	$K_{NV} = \frac{EJs^2}{2\bar{\alpha}\bar{\beta}L^3} [(s^2 - 2r^2)\bar{\beta}\Phi_1 - (s^2 - 2r^2)\bar{\alpha}\Phi_3]$	$K_{N\varphi} = K_{MV}$	$K_{NM} = K_{\varphi V}$	$K_{NN} = K_{VV}$

yields the following expressions for the integration constants C_1, C_2, C_3, C_4:

$$\left.\begin{aligned}
C_1 &= \frac{1}{2\bar{\alpha}\,\bar{\beta}s^2}\left[s^2\bar{\beta}L\varphi_0 + \bar{\beta}\,\frac{L^3}{EJ}\,N_0\right],\\
C_2 &= V_0,\\
C_3 &= \frac{1}{2\bar{\alpha}\,\bar{\beta}s^2}\left[s^2\bar{\alpha}L\varphi_0 - \bar{\alpha}\,\frac{L^3}{EJ}\,N_0\right],\\
C_4 &= -\frac{1}{2\bar{\alpha}\,\bar{\beta}}\left[r^2V_0 + \frac{L^2}{EJ}\,M_0\right].
\end{aligned}\right\} \qquad (3.16)$$

Substituting these values in (3.14) and (3.15), we can express $V(\eta)$, $\varphi(\eta)$, $M(\eta)$, $N(\eta)$ through the initial parameters V_0, φ_0, M_0, N_0 and the influence functions $K_{VV}, K_{V\varphi}, \ldots, K_{NN}$, given in Table 4.

The influence functions for the two other cases:

$$s = r \text{ and } s < r,$$

can be similarly determined.

§ 4. INFINITELY LONG BEAM

1

Consider an infinitely long beam. If a concentrated force P (Figure 41) acts at the origin of coordinates, we obtain the following homogeneous differential equation [for all points except the origin]:

$$\frac{d^4V}{d\eta^4} - 2r^2\frac{d^2V}{d\eta^2} + s^4V = 0. \qquad \text{[cf. (2.1)]} \qquad (4.1)$$

Assuming $s > r$, the general integral of (4.1) has the form

$$V(\eta) = C_1e^{-\bar{\alpha}\eta}\sin\bar{\beta}\eta + C_2e^{-\bar{\alpha}\eta}\cos\bar{\beta}\eta + C_3e^{\bar{\alpha}\eta}\sin\bar{\beta}\eta + C_4e^{\bar{\alpha}\eta}\cos\bar{\beta}\eta, \qquad (4.2)$$

where $\bar{\alpha}$ and $\bar{\beta}$ are given by (2.5).

For reasons of symmetry we consider only that part of the beam for which $x > 0$.

Since for $\eta \to \infty$ $\qquad V \to 0,$

we obtain $\qquad C_3 = C_4 = 0. \qquad (4.3)$

Substitution of (4.3) in (4.2) yields:

$$V(\eta) = C_1F_1 + C_2F_2, \qquad (4.4)$$

where

$$\left.\begin{aligned}
F_1 &= e^{-\bar{\alpha}\eta}\sin\bar{\beta}\eta,\\
F_2 &= e^{-\bar{\alpha}\eta}\cos\bar{\beta}\eta.
\end{aligned}\right\} \qquad (4.5)$$

The integration constants C_1 and C_2 can be determined from the conditions at the origin:

$$\text{at } \eta=0 \qquad \left.\begin{aligned} \varphi(0) &= \frac{1}{L}\frac{dV}{d\eta} = 0, \\ N(0) &= -\frac{EJ}{L^3}\left[\frac{d^3V}{d\eta^3} - 2r^2\frac{dV}{d\eta}\right] = -\frac{P}{2}, \end{aligned}\right\} \tag{4.6}$$

where $\varphi(0)$ is the slope and $N(0)$ the generalized shearing force for $\eta=0$.*

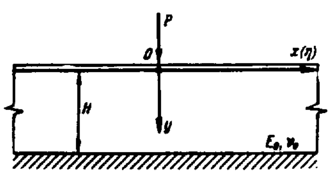


FIGURE 41.

Substitution of (4.4) in (4.6) yields:

$$\left.\begin{aligned} C_1F_1'(0) + C_2F_2'(0) &= 0, \\ C_1F_1'''(0) + C_2F_2'''(0) &= \frac{PL^3}{2EJ}. \end{aligned}\right\} \tag{4.7}$$

By solving (4.7) we obtain the integration constants:

$$\left.\begin{aligned} C_1 &= \frac{PL^3}{2EJ}\,\frac{F_2'(0)}{F_1'''(0)F_2'(0) - F_2'''(0)F_1'(0)}, \\ C_2 &= \frac{-PL^3}{2EJ}\,\frac{F_1'(0)}{F_1'''(0)F_2'(0) - F_2'''(0)F_1'(0)}, \end{aligned}\right\} \tag{4.8}$$

where $F_1'(0)$, $F_1'''(0)$, $F_2'(0)$, $F_2'''(0)$ are the first and third derivatives of F_1, and F_2 at $\eta=0$.

By substituting (4.8) in (4.4) and taking (1.7), (1.9), and (2.5) into account we obtain finally:

$$V(\eta) = \frac{P(1-\nu_0^2)}{E_0 b}\,\frac{1}{2\bar{\alpha}\bar{\beta}s^3}\,[\bar{\alpha}F_1(\eta) + \bar{\beta}F_2(\eta)]. \tag{4.9}$$

Expressions (1.11) and (1.12) for the bending moments and shearing forces respectively then become:

$$M(\eta) = \frac{PL}{4}\left[\frac{F_2(\eta)}{\bar{\alpha}} - \frac{F_1(\eta)}{\bar{\beta}}\right], \tag{4.10}$$

$$Q(\eta) = -\frac{P}{2}\left[F_2(\eta) - \frac{r^2}{2\bar{\alpha}\bar{\beta}}F_1(\eta)\right]. \tag{4.11}$$

* The following ordinary boundary conditions correspond to the generalized conditions (4.6):

$$\varphi(0) = \frac{1}{L}\frac{dV}{d\eta} = 0,$$
$$Q(0) = -\frac{EJ}{L^3}\frac{d^3V}{d\eta^3} = -\frac{P}{2}.$$

The reactions of the elastic foundation are, by (1.4),

$$q(\eta)=\frac{P}{4\bar{\alpha}\,\bar{\beta}L}\left[\bar{\beta}(s^2+2r^2)F_2(\eta)+\bar{\alpha}(s^2-2r^2)F_1(\eta)\right]. \tag{4.12}$$

Expressions (4.9) through (4.12) make possible rapid calculation of the stresses and strains in an infinite beam [on an elastic foundation]. They can be written in the following concise form, similar to that for an [ordinary] infinite beam (cf., for instance, /25/):

$$\left.\begin{aligned} V(\eta)&=\frac{P(1-\nu_0^2)}{E_0\delta}\,\bar{v}(\eta), & M(\eta)&=PL\bar{m}(\eta),\\ Q(\eta)&=-P\bar{Q}(\eta), & q(\eta)&=\frac{P}{L}\bar{q}(\eta), \end{aligned}\right\} \tag{4.13}$$

where

$$\left.\begin{aligned} \bar{v}(\eta)&=\frac{1}{2\bar{\alpha}\,\bar{\beta}s^2}\,[\bar{\alpha}F_1(\eta)+\bar{\beta}F_2(\eta)],\\ \bar{m}(\eta)&=\frac{1}{4}\left[\frac{F_2(\eta)}{\bar{\alpha}}-\frac{F_1(\eta)}{\bar{\beta}}\right],\\ \bar{Q}(\eta)&=\frac{1}{2}\left[F_2(\eta)-\frac{r^2}{2\bar{\alpha}\,\bar{\beta}}F_1(\eta)\right],\\ \bar{q}(\eta)&=\frac{1}{4\bar{\alpha}\,\bar{\beta}}\,[\bar{\beta}(s^2+2r^2)F_2(\eta)+\bar{\alpha}(s^2-2r^2)F_1(\eta)]. \end{aligned}\right| \tag{4.14}$$

The following relationships exist between these functions:

$$\left.\begin{aligned} \bar{v}'(\eta)&=-\bar{\varphi}(\eta)^{*}, & \bar{m}'(\eta)&=-\bar{Q}(\eta),\\ \bar{\varphi}'(\eta)&=\bar{m}(\eta), & \bar{Q}'(\eta)&=-\bar{q}(\eta). \end{aligned}\right\} \tag{4.15}$$

2

Expressions (4.9) through (4.12), or (4.13), are quite general. They are valid for any function $\psi(y)$, the accuracy being equal to that with which k, t, s^2, r^2, $\bar{\alpha}$, $\bar{\beta}$ have been determined.

Consider the case for which:

$$\psi(y)=\frac{\operatorname{sh}\gamma\dfrac{H-y}{L}}{\operatorname{sh}\gamma\dfrac{H}{L}}, \tag{4.16}$$

where L is defined by (1.7), and γ is a coefficient depending on the properties of the elastic foundation.

* $\bar{\varphi}(\eta)=\dfrac{1}{2\bar{\alpha}\,\bar{\beta}}F_1(\eta)$ is the dimensionless slope whose dimensional value is: $\varphi(\eta)=-\dfrac{P(1-\nu_0^2)}{E_0\delta L}\bar{\varphi}(\eta)$.

The constants entering in (4.9) through (4.12) are:

$$\left.\begin{aligned}
&k=\frac{E_0\delta}{H\left(1-\nu_0^2\right)}\psi_k, \qquad t=\frac{E_0\delta H}{12\left(1+\nu_0\right)}\psi_t,\\
&s^4=2\frac{L}{H}\psi_k, \qquad r^2=2\frac{1-\nu_0}{12}\frac{H}{L}\psi_t,\\
&\bar{\alpha}=\sqrt{\frac{s^2+r^2}{2}}, \qquad \bar{\beta}=\sqrt{\frac{s^2-r^2}{2}},\\
&\psi_k=\frac{1}{2}\frac{\gamma H}{L}\frac{\operatorname{sh}\frac{\gamma H}{L}\operatorname{ch}\frac{\gamma H}{L}+\frac{\gamma H}{L}}{\operatorname{sh}^2\frac{\gamma H}{L}},\\
&\psi_t=\frac{3}{2}\frac{L}{\gamma H}\frac{\operatorname{sh}\frac{\gamma H}{L}\operatorname{ch}\frac{\gamma H}{L}-\frac{\gamma H}{L}}{\operatorname{sh}^3\frac{\gamma H}{L}}.
\end{aligned}\right\} \tag{4.17}$$

[cf. (2.5) of this chapter and (3.18), (3.19) of Chapter I]

If the elastic foundation is a semi-infinite plane ($H\to\infty$), expressions (4.17) reduce to:

$$\left.\begin{aligned}
&k=\frac{E_0\delta}{2\left(1-\nu_0^2\right)}\frac{\gamma}{L},\\
&t=\frac{E_0\delta}{8\left(1+\nu_0\right)}\frac{L}{\gamma},\\
&s^4=\gamma,\\
&r^2=\frac{1-\nu_0}{4\gamma}.
\end{aligned}\right\} \tag{4.18}$$

Curves of $\bar{v}$, $\bar{m}$, $\bar{Q}$ and $\bar{q}$, calculated from (4.14) and (4.18) for $H\to\infty$, $\nu_0 = 0.3$, and $\gamma = 1.0$, $\gamma = 1.5$, are shown in Figures 42, 43, 44, 45. The abscissae are the dimensionless distances $\eta=\frac{x}{L}$, measured from the origin.

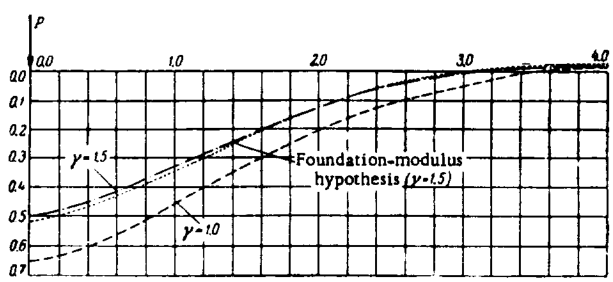
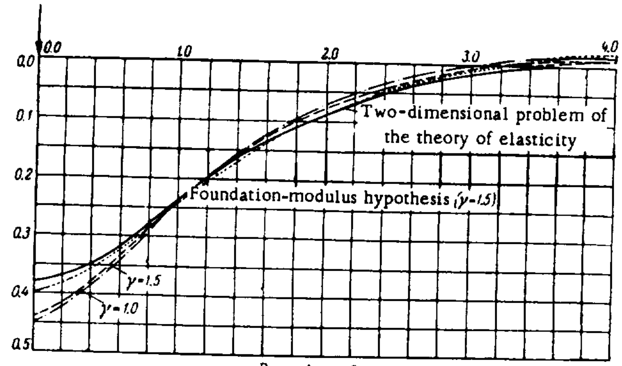


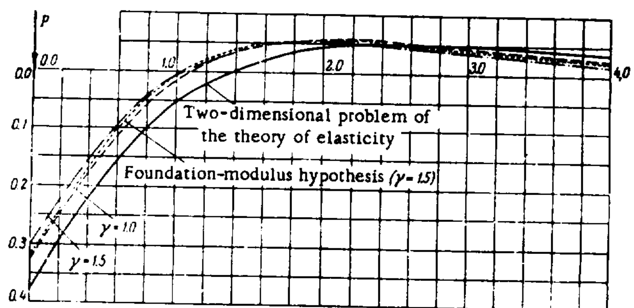
FIGURE 42.

These curves have been plotted only for positive values of η since [obviously] $\bar{v}$, $\bar{m}$, and $\bar{q}$ are even functions of η, only $\bar{Q}$ being an odd function. The dimensional functions V, M, Q and q are obtained from (4.13).

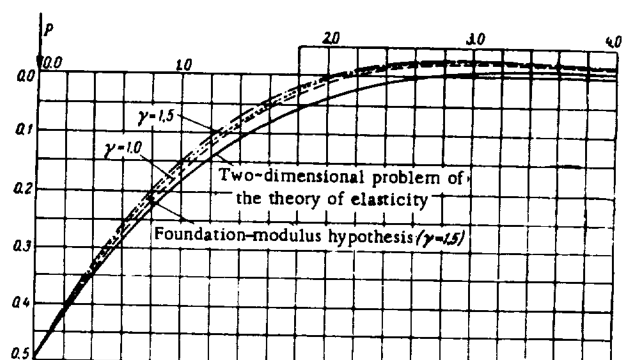
When γ increases, the bending moment at $\eta = 0$ decreases. This is due to the larger reactions near the point of action of the force. In addition, the deflections of the beam are considerably less.



Reactions $\bar{q}$
FIGURE 43.



Bending moments $\bar{m}$
FIGURE 44.



Shearing forces $\bar{Q}$
FIGURE 45.

Curves obtained by Gersevanov and Macheret (cf. /21, 25/) for an infinite beam on an elastic semi-infinite plane*, and also those obtained by postulating a foundation modulus, have been plotted in Figures 43, 44, and 45 for comparison. The foundation modulus is assumed to be:

$$k = \frac{E_0 \delta}{2(1-\nu_0^2)} \frac{\gamma}{L} \tag{4.19}$$

where

$$\gamma = 1.5; \qquad \nu_0 = 0.3, \qquad L = \sqrt[3]{\frac{2EJ(1-\nu_0^2)}{\delta E_0}}.$$

Comparison with the results, obtained when the foundation modulus is postulated, shows (see the curve for $\gamma = 1.5$ in Figure 44) that when allowance is made for shearing stresses in the elastic foundation, the maximum bending moment decreases. The difference is of the order of 4 % for the values of γ considered.

When $\psi(y)$ is given by (4.16), the solution presented here gives for $\gamma = 1.0$ and 1.5 lower absolute values of the maximum bending moment than the solution of the two-dimensional problem of the theory of elasticity. The difference is of the order of 15 to 20%, becoming less when γ decreases.

3

Consider an infinite beam on which a positive moment M_0 acts in a clockwise direction (Figure 46).

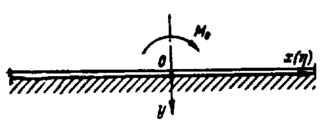


FIGURE 46.

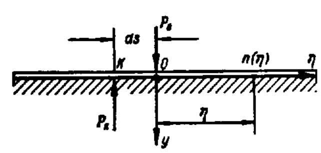


FIGURE 47.

Let two equal and opposite forces P be applied at points O and K situated at a distance ds from each other (Figure 47). We shall determine the deflections of the beam due to this couple. The first equation (4.13) becomes:

$$V_{P_O}(\eta) = \frac{P_O(1-\nu_0^2)}{E_0\delta}\bar{v}(\eta),$$

$$V_{P_K}(\eta) = -\frac{P_K(1-\nu_0^2)}{E_0\delta}\bar{v}(\eta + ds).$$

The total deflection of the beam at point $n(\eta)$ is:

$$V(\eta) = V_{P_O} + V_{P_K} = -\frac{P(1-\nu_0^2)}{E_0\delta}[\bar{v}(\eta + ds) - \bar{v}(\eta)]. \tag{4.20}$$

* This was not done in Figure 42, since the vertical displacements cannot be determined in the two-dimensional problem of the theory of elasticity.

Let ds tend to zero and P to infinity so that the product $Pds = M_0$ remains finite and constant. By multiplying and dividing the right side of (4.20) by $ds\,[= L d\eta]$, we obtain in the limit:

$$V(\eta) = -\frac{M_0(1-\nu_0^2)}{E_0\delta L}\bar{v}'(\eta) = \frac{M_0(1-\nu_0^2)}{E_0\delta L}\bar{v}_M(\eta). \tag{4.21}$$

Similarly:

$$\left.\begin{aligned} \varphi(\eta) &= \frac{M_0(1-\nu_0^2)}{E_0\delta L^2}\bar{\varphi}'(\eta) = \frac{M_0(1-\nu_0^2)}{E_0\delta L^2}\bar{\varphi}_M(\eta), \\ M(\eta) &= -M_0\bar{m}'(\eta) = M_0\bar{m}_M(\eta), \\ Q(\eta) &= \frac{M_0}{L}\bar{Q}'(\eta) = -\frac{M}{L}\bar{Q}_M(\eta), \\ q(\eta) &= -\frac{M_0}{L^2}\bar{q}'(\eta) = \frac{M_0}{L^2}\bar{q}_M(\eta). \end{aligned}\right\} \tag{4.22}$$

By (4.15) :

$$\left.\begin{aligned} \bar{v}_M(\eta) &= \bar{\varphi}(\eta), & \bar{m}_M(\eta) &= \bar{Q}(\eta), \\ \bar{\varphi}_M(\eta) &= \bar{m}(\eta), & \bar{Q}_M(\eta) &= \bar{q}(\eta). \end{aligned}\right\} \tag{4.23}$$

By differentiating the last equation (4.14) we obtain:

$$\bar{q}_M(\eta) = r^2F_2(\eta) + \frac{s^4 - 2r^4}{4\bar{\alpha}\,\bar{\beta}}F_1(\eta). \tag{4.24}$$

§ 5. RIGID BEAM

The case of an infinitely rigid beam is very important in the theory of beams of finite length. Analysis of such beams reduces to finding the reactions $q(x)$; the other unknowns can be found by means of the ordinary equations of statics.

It is convenient to resolve the external load into symmetrical and antisymmetrical components and to carry out the calculations separately. The final result is then obtained by superposition.

1. Symmetrical loading

The deflection of a rigid beam under the action of a symmetrical load is constant;

$$V(x) = C_0. \tag{5.1}$$

The vertical displacements of the surface of the elastic foundation are also constant beneath the beam, as follows from (1.4):

$$q(x) = kC_0. \tag{5.2}$$

The reactions of the elastic foundation are thus determined in a manner similar to Winkler's method. The only difference is that in our case $V(x)$ has no discontinuity at the beam ends, as would happen if the foundation modulus were postulated. In other words, the elastic single-layer foundation is strained even beyond the edges of the beam (Figure 48).

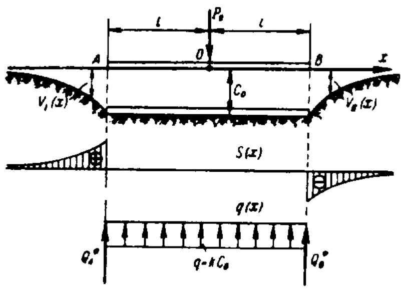


FIGURE 48.

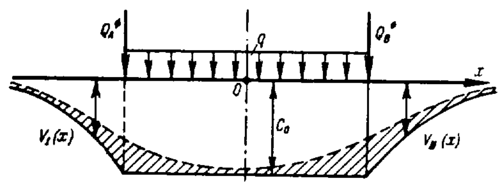


FIGURE 49.

In section 3 of Chapter I we introduced the generalized shearing force:

$$S(x) = \int_0^H \tau_{yx}\psi(y)\,dF = 2tV'(x) \tag{5.3}$$

[cf. (3.10) of Chapter I]

which is discontinuous at points where concentrated forces act at the surface of the elastic foundation.

In accordance with (5.1), S becomes zero beneath the beam. Beyond the edges of the beam, the generalized shearing force is, however, different from zero, so that $S(x)$ has discontinuities at $x=-l$ and $x=l$. Hence, concentrated reactions Q^{Φ} arise at the beam ends, which are due to stresses in the elastic foundation beyond the beam edges.

The existence of concentrated reactions Q^{Φ} can be proved also by different reasoning. The assumption that only distributed reactions, given by (5.2), act on the bottom of the rigid beam leads to a contradiction: the vertical displacements of the surface of a single-layer foundation acted upon by a uniformly distributed load are not constant (cf. (3.39) Chapter I). In order that condition (5.1) be satisfied beneath the beam, concentrated forces must act at the beam ends, causing additional displacements of the foundation surface (the hatched part of the displacement diagram in Figure 49).

In the general case the concentrated reactions Q^{Φ} are equal to the difference between the values of S to the left and to the right of the beam end:

$$\left.\begin{aligned} Q_A^{\Phi} &= S_0(-l) - S_b(-l),\\ Q_B^{\Phi} &= S_b(l) - S_0(l), \end{aligned}\right\} \tag{5.4}$$

where S_0 = generalized shearing force acting in the free foundation, S_b = generalized shearing force acting in the foundation beneath the beam.

The sign of Q^{Φ} is determined in a similar manner as the sign of the reactions $q(x)$, being positive for forces acting upward.

The stresses in the elastic foundation beyond the beam edges are given by:

$$-2tV'' + kV = 0. \tag{5.5}$$

Solving (5.5) for V, we obtain:

$$\left.\begin{array}{ll} \text{at } x \leqslant -l & V_{\mathrm{I}} = C_0 e^{\alpha(x+l)}, \\ \text{at } x \geqslant l & V_{\mathrm{II}} = C_0 e^{-\alpha(x-l)}. \end{array}\right\} \tag{5.6}$$

Taking (5.1), (5.6), and (5.3) into account, we obtain from (5.4):

$$Q_A^{\Phi} = Q_B^{\Phi} = 2\alpha t C_0, \tag{5.7}$$

where

$$\alpha = \sqrt{\frac{k}{2t}}. \tag{5.8}$$

The displacement C_0 is found from the equilibrium condition of the beam by equating the projection of all vertical forces to zero. The forces acting on the beam are the known external load P_0, the uniform reaction q, and the two forces Q_A^{Φ} and Q_B^{Φ}; therefore [by (5.2)]

$$C_0 = \frac{P_0}{2(kl + 2\alpha t)}. \tag{5.9}$$

Substitution of (5.9) in (5.2) and (5.7) yields:

$$q = \frac{P_0}{2l}\,\frac{1}{1 + 2\,\frac{\alpha t}{kl}}, \tag{5.10}$$

$$Q_A^{\Phi} = Q_B^{\Phi} = \frac{P_0}{2}\,\frac{1}{1 + \frac{kl}{2\alpha t}}. \tag{5.11}$$

The constant C_0 could also have been determined from the generalized (variational) equilibrium condition for the entire system (beam and elastic foundation), obtained by equating to zero the total work done by all external and internal forces in the system over the virtual displacement $\bar{v}(x, y) = 1 \cdot \psi(y)$:

$$-\int_{-\infty}^{+\infty}\int_{0}^{H} \sigma_y \psi'(y)\, dF\, dx + P_0\psi(0) = 0, \tag{5.12}$$

where $dF = \delta\, dy$, and σ_y is the normal stress, given by the first equation (3.3) of Chapter I.

Substituting (3.3) of Chapter I in (5.12), taking into account (1.3), (5.1), (5.6), and noting that $\psi(0) = 1$, we obtain:

$$C_0 k\left[\int_{-\infty}^{-l} e^{\alpha(x+l)}\, dx + \int_{-l}^{+l} dx + \int_{l}^{\infty} e^{-\alpha(x-l)}\, dx\right] = P_0, \tag{5.12'}$$

or

$$C_0 k\left[2l + \frac{2}{\alpha}\right] = 2C_0\,[kl + 2\alpha t[= P_0, \tag{5.12''}$$

from which we again obtain (5.9).

From (5.12') or (5.12'') and (5.7) it is seen that each reaction Q^Φ is equal in magnitude to k times the volume of the displacements in the elastic foundation beyond the corresponding beam end:

$$\left.\begin{aligned} Q_A^\Phi &= k\int_{-\infty}^{-l} C_0 e^{\alpha(x+l)}\,dx = 2\alpha t C_0,\\ Q_B^\Phi &= k\int_{l}^{\infty} C_0 e^{-\alpha(x-l)}\,dx = 2\alpha t C_0. \end{aligned}\right\} \tag{5.7'}$$

A similar result is obtained if the displacements given by (5.1) and (5.6) for $C_0 = 1$ are considered as virtual displacements. Unlike (5.12), the work done by the internal forces is in this case the sum of the work done by the normal stresses σ_y and by the shearing stresses τ_{yx}. The final result will again be given by (5.12'').

2. Antisymmetrical loading

The deflection of the beam due to an antisymmetrical load is:

$$V(x) = \theta_0 x, \tag{5.13}$$

where $\theta_0 = \operatorname{tg}\varphi_0$ = the slope of the beam (Figure 50).

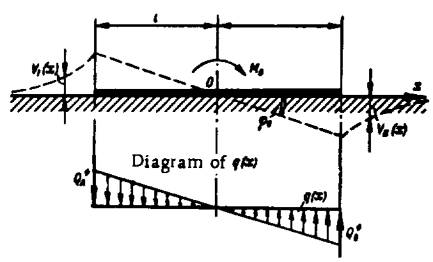


FIGURE 50.

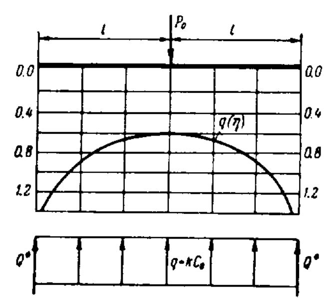


FIGURE 51.

It follows in this case from (1.4) that:

$$q(x) = k\theta_0 x. \tag{5.14}$$

In order to determine the concentrated reactions Q^Φ, we calculate the generalized shearing forces $S(x)$.

Proceeding from (5.3), and noting that the displacements of the foundation beyond the beam ends are:

$$\left.\begin{aligned} &\text{at } x \leqslant -l \qquad V_{\mathrm{I}} = -\theta_0 l e^{\alpha(x+l)};\\ &\text{at } x \geqslant l \qquad V_{\mathrm{II}} = \theta_0 l e^{-\alpha(x-l)}, \end{aligned}\right\} \tag{5.15}$$

we obtain the following expressions for the generalized shearing force:

$$\left.\begin{aligned}&\text{at } x \leqslant -l && S = -2\alpha t\theta_0 l e^{\alpha(x+l)};\\&\text{at } -l < x < l && S = 2t\theta_0;\\&\text{at } x \geqslant l && S = -2\alpha t\theta_0 l e^{-\alpha(x-l)}.\end{aligned}\right\} \quad (5.16)$$

The concentrated reactions at the beam ends can now be determined from (5.4):

$$-Q_A^{\Phi} = Q_B^{\Phi} = 2t(1+\alpha l)\theta_0, \quad (5.17)$$

where

$$\alpha = \sqrt{\frac{k}{2t}}.$$

From the equilibrium condition for the beam, obtained by equating to zero the sum of all moments acting on the beam about its center, we find:

$$\theta_0 = \frac{3M_0}{2l\,[kl^2 + 6t(1+\alpha l)]}. \quad (5.18)$$

Substitution of (5.18) in (5.14) and (5.17) leads to the following expressions for the reactions:

$$q = \frac{3M_0}{2l^3\left[1 + 6\frac{t}{l^2 k}(1+\alpha l)\right]}x, \quad (5.19)$$

$$Q^{\Phi} = \frac{M_0}{2l\left[1 + \frac{kl^2}{6t(1+\alpha l)}\right]}, \quad (5.20)$$

where M_0 is the sum of the external moments about the coordinate origin.

Expressions (5.10), (5.11) (for symmetrical loading), and (5.19), (5.20) (for antisymmetrical loading), give the complete solution for a rigid beam, since the bending moments and shearing forces can be determined by known methods once the reactions have been determined.

3. Calculation examples and analysis of results

The exact solution of the problem of a plane symmetrically loaded punch, obtained by Sadovskii, is:

$$q(\eta) = \frac{P_0}{\pi l\sqrt{1-\eta^2}}, \quad (5.21)$$

where $\eta = \frac{x}{l}$ (Figure 51).

The reactions given by (5.21) are also shown in Figure 51. It can be seen that they increase to infinity toward the punch ends.

The elastic foundation is not considered by us to be a semi-infinite plane but a single-layer model whose properties are determined by two parameters k and t, in which concentrated forces Q^{Φ} correspond to the infinitely large reactions $q(\eta)$ obtained for the exact solution.

The concentrated reactions Q^{Φ} are obviously not real forces acting in the elastic foundation at the beam ends. Their appearance is due to the stresses in the elastic foundation beyond the punch ends. They should therefore be considered as fictitious forces by which allowance is made for the influence of the free foundation on the stresses in the punch.

We put:

$$\psi(y)=\frac{\operatorname{sh}\gamma\frac{H-y}{l}}{\operatorname{sh}\frac{\gamma H}{l}}, \tag{5.22}$$

where l is the half-length of the beam, and γ a coefficient characterizing the variation with depth of the vertical displacements in the foundation.

In accordance with (1.3) and (5.22), the elastic parameters of the foundation are:

$$\left.\begin{aligned} k&=\frac{E_0\delta}{H(1-\nu_0^2)}\psi_k, \quad t=\frac{E_0\delta H}{12(1+\nu_0)}\psi_t,\\ \alpha&=\frac{1}{H}\frac{\sqrt{6(1-\nu_0)}}{1-\nu_0}\psi_\alpha, \end{aligned}\right\} \tag{5.23}$$

where

$$\left.\begin{aligned} \psi_k&=\frac{1}{2}\frac{\gamma H}{l}\frac{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}+\frac{\gamma H}{l}}{\operatorname{sh}^2\frac{\gamma H}{l}},\\ \psi_t&=\frac{3}{2}\frac{l}{\gamma H}\frac{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}-\frac{\gamma H}{l}}{\operatorname{sh}^2\frac{\gamma H}{l}},\\ \psi_\alpha&=\frac{\gamma H}{l}\sqrt{\frac{1}{3}\frac{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}+\frac{\gamma H}{l}}{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}-\frac{\gamma H}{l}}}. \end{aligned}\right\} \tag{5.24}$$

[cf. (3.18), (3.19), (3.32) of Chapter I]

Substitution of (5.23) in (5.10), (5.11), (5.19), and (5.20) yields:

for the symmetrical load,

$$\left.\begin{aligned} q&=\frac{P_0}{2l}\frac{1}{1+\frac{\sqrt{2(1-\nu_0)}}{2\gamma}\sqrt{\frac{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}-\frac{\gamma H}{l}}{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}+\frac{\gamma H}{l}}}},\\ Q^{\Phi}&=\frac{P_0}{2}\frac{1}{1+\frac{2\gamma}{\sqrt{2(1-\nu_0)}}\sqrt{\frac{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}+\frac{\gamma H}{l}}{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}-\frac{\gamma H}{l}}}}; \end{aligned}\right\} \tag{5.25}$$

for the antisymmetrical load,

$$
\left.
\begin{aligned}
q &= \frac{M_0}{l^3}\,\frac{1}{\dfrac{2}{3}+\dfrac{1-\nu_0}{\gamma^2}\,\dfrac{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}-\frac{\gamma H}{l}}{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}+\frac{\gamma H}{l}}+\dfrac{\sqrt{2(1-\nu_0)}}{\gamma}\sqrt{\dfrac{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}-\frac{\gamma H}{l}}{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}+\frac{\gamma H}{l}}}}\,x,\\
Q^{\Phi} &= \frac{M_0}{2l}\,\frac{1+\gamma\dfrac{\sqrt{2(1-\nu_0)}}{1-\nu_0}\sqrt{\dfrac{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}+\frac{\gamma H}{l}}{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}-\frac{\gamma H}{l}}}}{1+\dfrac{2}{3}\,\dfrac{\gamma^2}{1-\nu_0}\,\dfrac{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}+\frac{\gamma H}{l}}{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}-\frac{\gamma H}{l}}+\gamma\dfrac{\sqrt{2(1-\nu_0)}}{1-\nu_0}\sqrt{\dfrac{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}+\frac{\gamma H}{l}}{\operatorname{sh}\frac{\gamma H}{l}\operatorname{ch}\frac{\gamma H}{l}-\frac{\gamma H}{l}}}}.
\end{aligned}
\right\}
\tag{5.26}
$$

Consider the case of symmetrical loading, for which curves of the dimensionless magnitudes $\bar{q}$ and $\bar{Q}^{\Phi}$ are given in Figures 52 and 53. for $\nu_0 = 0.3$ and $\gamma=1.0$; $\gamma=1.5$; $\gamma=2.0$. The dimensionless reactions q and Q^{Φ} are obtained from:

$$q = \frac{P_0}{2l}\bar{q}, \quad Q^{\Phi} = \frac{P_0}{2}\bar{Q}^{\Phi}. \tag{5.27}$$

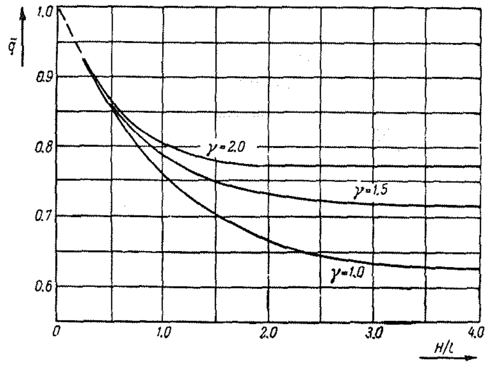


FIGURE 52.

It can be seen that for ratios of $\frac{H}{l}$ above 1.5 to 2.0 the reactions remain constant for the values of γ considered. When $\frac{H}{l}$ exceeds this value, therefore, the elastic foundation behaves like an elastic semi-infinite plane $(H = \infty)$. It can also be seen that when γ increases, the concentrated reactions at the beam ends decrease, the foundation behaving more like Winkler's model.

In order to compare (5.25) with the exact solution given by the theory of elasticity, we determined the bending moment m_0 at the center of the beam, due to the reactions q [(5.10)] and Q^{Φ} [(5.11)] alone:

$$m_0 = \frac{ql^2}{2} + Q^{\Phi}l = \frac{P_0 l}{4}\,\frac{kl + 4\alpha t}{kl + 2\alpha t}. \tag{5.28}$$

Insertion of (5.23) and (5.24) yields:

$$m_0 = \frac{P_0 l}{2} \frac{3\gamma + \sqrt{6(1-\nu_0)} \sqrt{3\,\frac{\operatorname{sh}\bar{\gamma}\operatorname{ch}\bar{\gamma} - \bar{\gamma}}{\operatorname{sh}\bar{\gamma}\operatorname{ch}\bar{\gamma} + \bar{\gamma}}}}{6\gamma + \sqrt{6(1-\nu_0)} \sqrt{3\,\frac{\operatorname{sh}\bar{\gamma}\operatorname{ch}\bar{\gamma} - \bar{\gamma}}{\operatorname{sh}\bar{\gamma}\operatorname{ch}\bar{\gamma} + \bar{\gamma}}}}, \tag{5.29}$$

with $\bar{\gamma} = \gamma \frac{H}{l}$.

In the case of an elastic semi-infinite plane $\left(\frac{H}{l} = \infty\right)$, (5.29) reduces to:

$$m_0 = P_0 l \bar{m}_0, \tag{5.30}$$

where

$$\bar{m}_0 = \frac{1}{2} \frac{\gamma + \sqrt{2(1-\nu_0)}}{2\gamma + \sqrt{2(1-\nu_0)}}. \tag{5.31}$$

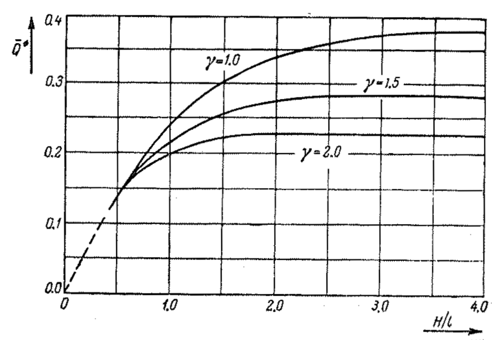


FIGURE 53.

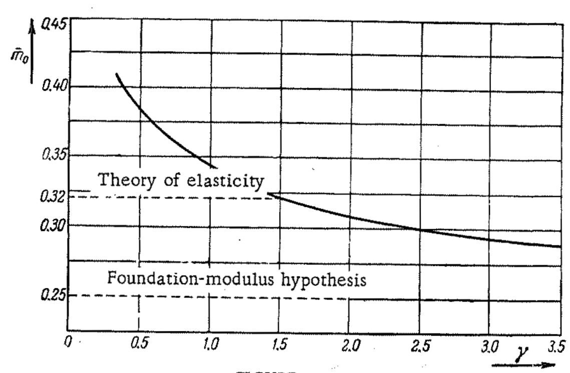


FIGURE 54.

The value of $\bar{m}_0$ given by (5.31) has been plotted in Figure 54 for $\nu_0 = 0.3$ as a function of γ. It is seen that $m_0 = 0.32$ for $\gamma = 1.5$, which is Gorbunov-Posadov's result for a rigid beam /26/. With increasing γ the value of $\bar{m}_0$ approaches 0.25 asymptotically, which is the solution when the foundation modulus is postulated.

The reactions thus obtained for $\gamma = 1.5$ are therefore equal to those obtained in the two-dimensional solution (5.21) for the moments in the center of the beam, due to these reactions. Since, for a rigid beam acted upon by a symmetrical load, q is uniquely determined by P_0 and l, the bending-moment diagram, obtained by the method proposed by us, will be similar to that obtained in the two-dimensional solution given by the theory of elasticity.

As an example, Figures 55 and 56 show curves of M and Q corresponding to the three most important cases of loading of a rigid beam for $\gamma = 1.5$. Comparison with the exact solution* (broken curves and numbers in parentheses) shows that the results differ only near the beam ends.

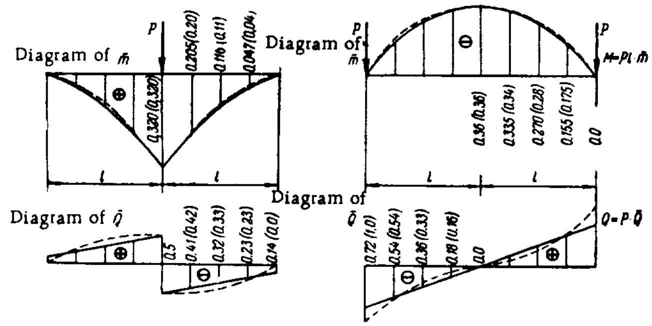


FIGURE 55.

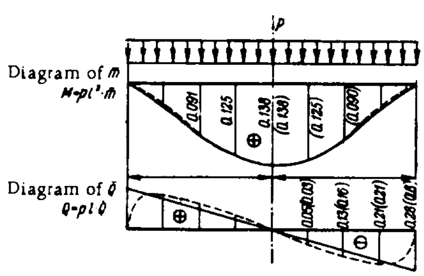


FIGURE 56.

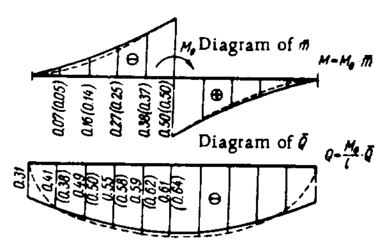


FIGURE 57.

A similar comparison for the case of antisymmetrical loading shows that the results obtained for $\gamma = 1.5$ are very similar (Figure 57).

* See Gorbunov-Posadov Raschet konstruktsii na uprugom osnovanii (Analysis of Structures on Elastic Foundations).—Gosstroiizdat, Part 1, Chapter 1, §6. 1953.

4. Allowance for plastic deformations beneath the beam ends

It was assumed until now that the vertical displacements $V(x)$ of the surface of the elastic foundation are continuous. This necessitated the assumption of concentrated reactions Q^{Φ} beneath the beam ends.

Under actual conditions these intense reactions cause plastic deformations beneath the beam ends, leading to considerable changes in the general stress pattern in beam and foundation. The methods of the theory of elasticity no longer apply to the soil in this case and a special analysis, in which allowance is made for the elastic-plastic deformations of the foundation, becomes necessary.

The approximative method proposed can, however, be applied to this case. Assume that plastic deformations have occurred beneath the ends of a symmetrically loaded beam, as a result of which $V(x)$ has discontinuities at $x = l$ and $x = -l$, equal to:

$$c = (1 - \beta) C_0, \qquad (5.32)$$

where β is a coefficient characterizing the settling tendency of the elastic foundation (Figure 58).

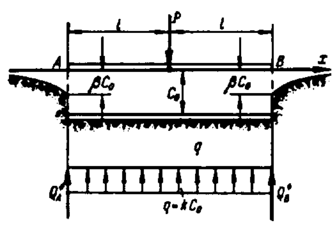


FIGURE 58.

As before, the reactions consist of the uniform reaction q and the fictitious forces Q^{Φ}, where by (5.2)

$$q = kC_0. \qquad (5.33)$$

According to (5.3) and (5.4):

$$Q_A^{\Phi} = Q_B^{\Phi} = 2\alpha t\beta C_0 \qquad (5.34)$$

From the equilibrium condition of the beam we find:

$$C_0 = \frac{P_0}{2(kl + 2\alpha\beta t)}. \qquad \text{[cf. (5.9)]} \qquad (5.35)$$

Substitution of (5.35) in (5.33) and (5.34) yields:

$$\left.\begin{aligned} q &= \frac{P_0}{2l}\frac{1}{1 + 2\frac{\alpha\beta t}{kl}}, \\ Q^{\Phi} &= \frac{P_0}{l}\frac{1}{1 + \frac{kl}{2\alpha\beta t}}. \end{aligned}\right\} \qquad (5.36)$$

[cf. (5.10), (5.11)]

When $\beta = 0$, the elastic foundation is not strained beyond the beam ends; this corresponds to Winkler's model. For $\beta = 1$, (5.35) becomes identical with (5.9).

Similar considerations apply to the case of antisymmetrical loading.

§ 6. ELASTIC BEAM OF FINITE LENGTH

1

The differential equation of the bending of a beam on a single-layer elastic foundation is (cf. (1.8)):

$$\frac{d^4V}{d\eta^4} - 2r^2\frac{d^2V}{d\eta^2} + s^4V = \frac{pL^4}{EJ}. \tag{6.1}$$

Here $\eta = \frac{x}{L}$ and

$$L = \sqrt[3]{\frac{2EJ(1-\nu_0^2)}{E_0\delta}}, \tag{6.2}$$

where δ = beam width.

The coefficients in (6.1) are [cf. (1.9)]:

$$\left.\begin{aligned} r^2 &= \frac{1}{2}\frac{1-\nu_0}{L}\int_0^H \psi_1^2\,dy, \\ s^4 &= 2L\int_0^H \psi_1'^2\,dy. \end{aligned}\right\} \tag{6.3}$$

The general solution of the problem can be presented in the form [cf. (3.10)]:

$$\left.\begin{aligned} V(\eta) &= V_0K_{VV} + \varphi_0K_{V\varphi} + M_0K_{VM} + N_0K_{VN} - F_V, \\ \varphi(\eta) &= V_0K_{\varphi V} + \varphi_0K_{\varphi\varphi} + M_0K_{\varphi M} + N_0K_{\varphi N} - F_\varphi, \\ M(\eta) &= V_0K_{MV} + \varphi_0K_{M\varphi} + M_0K_{MM} + N_0K_{MN} - F_M, \\ N(\eta) &= V_0K_{NV} + \varphi_0K_{N\varphi} + M_0K_{NM} + N_0K_{NN} - F_N, \end{aligned}\right\} \tag{6.4}$$

where $K_{VV}, K_{V\varphi}, \ldots, K_{NM}, K_{NN}$ are the influence functions, given in Table 2; F_V, F_φ, F_M, F_N are functions depending on the external load (cf. e.g., (3.11)).

2.

To determine the initial parameters V_0, φ_0, M_0, N_0, it is necessary to consider the boundary conditions at the beam ends.

If the beam ends are prevented from moving downward, the boundary conditions can be written in the usual way:

a) for simply supported ends:

$$V = 0, \quad M = 0; \tag{6.5'}$$

b) for built-in ends:

$$V=0, \quad \varphi=0. \tag{6.5''}$$

If the beam ends are free to move downward, the compatibility conditions for the beam and the elastic foundation must be taken into account when establishing the boundary conditions. The vertical displacements $V(x)$ of the free parts (I) and (II) of the foundation (Figure 59) are determined except for the constants D_1 and D_2; the vertical displacements of the foundation beneath the beam are determined except for four constants C_1, C_2, C_3, C_4 (cf. (2.9)). Six independent conditions (three for each end of the beam) have therefore to be established in order to determine these constants.

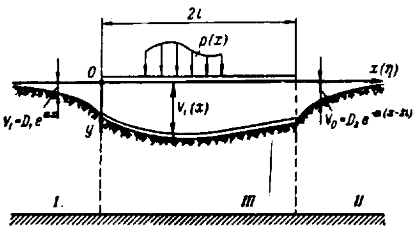


FIGURE 59.

An obvious condition for each end free of load is:

$$M=0. \tag{6.6'}$$

The two other conditions can be written as follows:

$$V_0=V_b; \tag{6.6''}$$
$$S=N \text{ at } x=0 \text{ (or } x=2l),$$

where V_0 = vertical displacement of foundation, V_b = vertical displacement of beam, S = generalized shearing force in free parts (I, II) of foundation, and N = generalized shearing force in the foundation beneath beam (part III).

We note that (6.6'') corresponds to the conditions for a rigid beam.

Only simply supported beams will be considered henceforth, since these are of the greatest practical interest while their analysis is the most complicated.

3

By placing the origin of coordinates at the left end of the beam (Figure 59) we find from (6.6) and (5.3):

$$M_0=0, \quad N_0=S(0)=2\alpha t V_0 \tag{6.7}$$

The system (6.4) now becomes:

$$\left.\begin{aligned} V(\eta) &= (K_{VV} + 2\alpha t K_{VN}) V_0 + K_{V\varphi}\varphi_0 - F_V, \\ \varphi(\eta) &= (K_{\varphi V} + 2\alpha t K_{\varphi N}) V_0 + K_{\varphi\varphi}\varphi_0 - F_\varphi, \\ M(\eta) &= (K_{MV} + 2\alpha t K_{MN}) V_0 + K_{M\varphi}\varphi_0 - F_M, \\ N(\eta) &= (K_{NV} + 2\alpha t K_{NN}) V_0 + K_{N\varphi}\varphi_0 - F_N. \end{aligned}\right\} \qquad (6.8)$$

The boundary conditions at the otner end of the beam are:

$$M\left(\frac{2l}{L}\right) = 0, \quad N\left(\frac{2l}{L}\right) = -2\alpha t V\left(\frac{2l}{L}\right), \qquad (6.9)$$

from which the other two initial parameters V_0 and φ_0 are obtained:

$$\left.\begin{aligned} V_0 &= \frac{a_1 F_M\left(\frac{2l}{L}\right) - a_2\left[F_N\left(\frac{2l}{L}\right) + 2\alpha t F_V\left(\frac{2l}{L}\right)\right]}{a_1^2 - a_2 a_3}, \\ \varphi_0 &= -\frac{a_3 F_M\left(\frac{2l}{L}\right) - a_1\left[F_N\left(\frac{2l}{L}\right) + 2\alpha t F_V\left(\frac{2l}{L}\right)\right]}{a_1^2 - a_2 a_3}. \end{aligned}\right\} \qquad (6.10)$$

The coefficients in (6.10) are:

$$\left.\begin{aligned} a_1 &= K_{MV}\left(\frac{2l}{L}\right) + 2\alpha t K_{MN}\left(\frac{2l}{L}\right), \\ a_2 &= K_{M\varphi}\left(\frac{2l}{L}\right), \\ a_3 &= K_{NV}\left(\frac{2l}{L}\right) + 4\alpha t K_{NN}\left(\frac{2l}{L}\right) + (2\alpha t)^2 K_{VN}\left(\frac{2l}{L}\right). \end{aligned}\right\} \qquad (6.11)$$

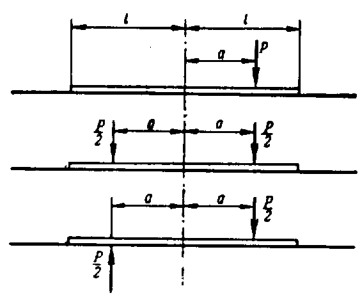


FIGURE 60.

Equations (6.8) and (6.10) represent the required solution when the bending of a beam of finite length, placed on an elastic foundation and acted upon by external loads, is considered.

When the beam is long, the origin of coordinates should be placed at the beam center, and the load should be resolved into symmetrical and antisymmetrical components (Figure 60); this increases the accuracy of the results. The method of solution is similar to that employed above.

In this case the following formulas are obtained:

a) for the symmetrical load:

$$\left.\begin{aligned} V(\eta) &= K_{VV}V_0 + K_{VM}M_0 - F_V, \\ \varphi(\eta) &= K_{\varphi V}V_0 + K_{\varphi M}M_0 - F_\varphi, \\ M(\eta) &= K_{MV}V_0 + K_{MM}M_0 - F_M, \\ N(\eta) &= K_{NV}V_0 + K_{NM}M_0 - F_N, \end{aligned}\right\} \tag{6.12}$$

where

$$V_0 = \frac{b_1 F_M\left(\frac{l}{L}\right) - b_2\left[F_N\left(\frac{l}{L}\right) + 2\alpha t F_V\right]}{b_1 b_4 - b_2 b_3},$$

$$M_0 = -\frac{b_3 F_M\left(\frac{l}{L}\right) - b_4\left[F_N\left(\frac{l}{L}\right) + 2\alpha t F_V\right]}{b_1 b_4 - b_2 b_3};$$

b) for the antisymmetrical load:

$$\left.\begin{aligned} V(\eta) &= K_{V\varphi}\varphi_0 + K_{VN}N_0 - F_V, \\ \varphi(\eta) &= K_{\varphi\varphi}\varphi_0 + K_{\varphi N}N_0 - F_\varphi, \\ M(\eta) &= K_{M\varphi}\varphi_0 + K_{MN}N_0 - F_M, \\ N(\eta) &= K_{N\varphi}\varphi_0 + K_{NN}N_0 - F_N, \end{aligned}\right\} \tag{6.13}$$

where

$$\varphi_0 = \frac{b_1 F_M\left(\frac{l}{L}\right) - b_2\left[F_N\left(\frac{l}{L}\right) + 2\alpha t F_V\left(\frac{l}{L}\right)\right]}{b_1 b_6 - b_2 b_5},$$

$$N_0 = -\frac{b_5 F_M\left(\frac{l}{L}\right) - b_6\left[F_N\left(\frac{l}{L}\right) + 2\alpha t F_V\left(\frac{l}{L}\right)\right]}{b_1 b_6 - b_2 b_5}.$$

The coefficients in (6.12) and (6.13) are:

$$\left.\begin{aligned} b_1 &= K_{NM}\left(\frac{l}{L}\right) + 2\alpha t K_{VM}\left(\frac{l}{L}\right), & b_4 &= K_{MV}\left(\frac{l}{L}\right), \\ b_2 &= K_{MM}\left(\frac{l}{L}\right), & b_5 &= K_{N\varphi}\left(\frac{l}{L}\right) + 2\alpha t K_{V\varphi}\left(\frac{l}{L}\right), \\ b_3 &= K_{NV}\left(\frac{l}{L}\right) + 2\alpha t K_{VV}\left(\frac{l}{L}\right), & b_6 &= K_{M\varphi}\left(\frac{l}{L}\right). \end{aligned}\right\} \tag{6.14}$$

Both (6.12) and (6.13) include only loads acting either to the right or to the left of the origin.

The equilibrium condition of the beam, the contact condition of equal vertical displacements of beam and foundation, and the continuity condition of foundation displacements are all satisfied by (6.8), (6.12), and (6.13). The statical boundary conditions are, however, only approximately satisfied: when the bending moments M vanish, the shearing forces Q at the beam ends differ from zero in the general case. This is explained by the properties of the model adopted for the elastic foundation, which is characterized by two generalized parameters k and t.

4

Analysis of these results shows that the elastic properties of both beam and foundation are characterized by a single magnitude:

$$\lambda = \frac{l}{L}, \tag{6.15}$$

which represents the reduced half-length of the beam.

In the two-dimensional problem of the theory of elasticity (solved, e.g., by the method of Gorbunov-Posadov), a magnitude called the index of beam flexibility is introduced as principal elastic characteristic, being defined as follows:

$$t = \frac{\pi E_0 \delta l^3}{4(1-\nu_0^2) EJ}, \tag{6.16}$$

where l = half-length of beam, δ = width of beam.

By comparing (6.2), (6.15), and (6.16) the following relationship between λ and t is obtained:

$$\lambda = \sqrt[3]{\frac{2t}{\pi}}. \tag{6.17}$$

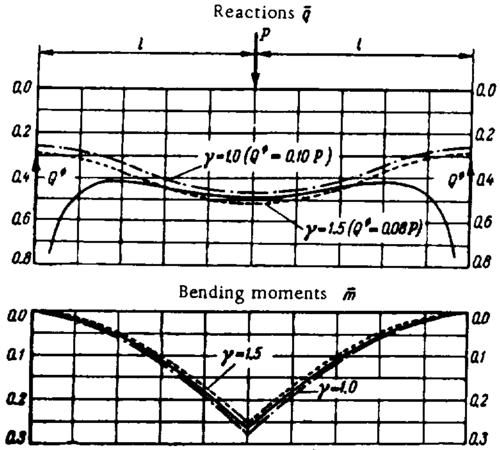
Several examples will now be discussed in order to show the effect of λ on the results. It will be assumed that the elastic foundation is a semi-infinite plane ($H = \infty$), and that $\psi(y)$ is given by (4.16).

The dimensionless reactions $\bar{q}$ and bending moments $\bar{m}$, calculated by (6.12) for two beams, each loaded at the center by a concentrated force P, are shown in Figures 61 and 62. The elastic characteristics of these beams are respectively: $\lambda = 1.24$ ($t = 3.0$), and $\lambda = 1.64$ ($t = 7.0$). The calculations were performed for $\gamma = 1.0$ and $\gamma = 1.5$, for $\nu_0 = 0$, and $H = \infty$. Curves of $\bar{q}$ and $\bar{m}$, plotted on the basis of Gorbunov-Posadov's data /26/ for $t = 3.0$ and $t = 7.0$, have been drawn for comparison in the same figures (full lines). The actual reactions and bending moments are respectively:

$$q = \bar{q}\frac{P}{l}, \quad M = \bar{m}Pl.$$

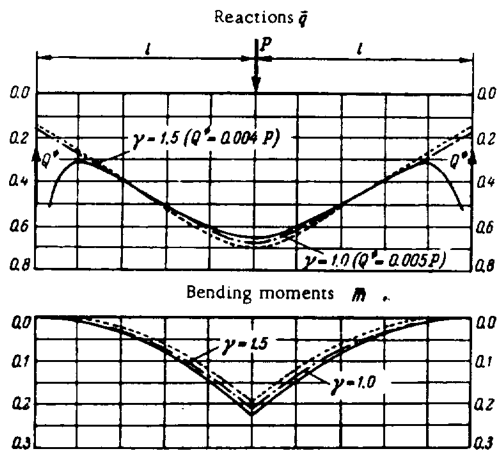
It is seen that the concentrated reactions Q^{Φ} at the beam ends vary inversely with λ, as do the bending moments. This variation is more pronounced when the value of λ is less. On the other hand, the more flexible or longer the beam, i.e., the larger the value of γ, the larger are the reactions near the point of application of the load, leading to a reduction in the bending moment at the beam center.

The dimensionless reactions $\bar{q}$ obtained by Gorbunov-Posadov's method are almost equal to those obtained by our method for $\gamma = 1.0$ and $\gamma = 1.5$ along the entire beam, except near its ends. As a result, satisfactory agreement is also observed for the corresponding bending-moment diagrams.



$\lambda=1.24$ $(t=3.0)$

FIGURE 61



$\lambda=1.64$ $(t=7.0)$

FIGURE 62

For $\gamma=1.5$, the bending moments for both finite elastic and infinite beams, determined by our method, are slightly less than those obtained when solving the two-dimensional problem of the theory of elasticity.

The dimensionless deflections and bending moments of a rigid beam (broken lines, numbers in parentheses) and of an elastic beam of reduced half-length $\lambda=0.86$ $(t=1)$, are shown in Figure 63 for $\gamma=1.5$, $\nu_0=0.3$, $H=\infty$. Comparison of these diagrams shows that (5.23) can be used for beams when $0<\lambda<0.86$. Such beams can be considered to be rigid.

A similar comparison, performed in Figure 64 for an elastic beam loaded by a concentrated force, and for an infinite beam, represented by broken lines and numbers in parentheses, shows that the results become practically identical for $\lambda=1.85$ $(t=10)$: the difference between the maximum bending moments is only 5 %*.

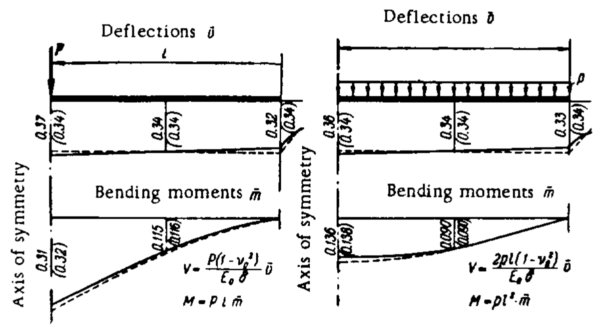


FIGURE 63.

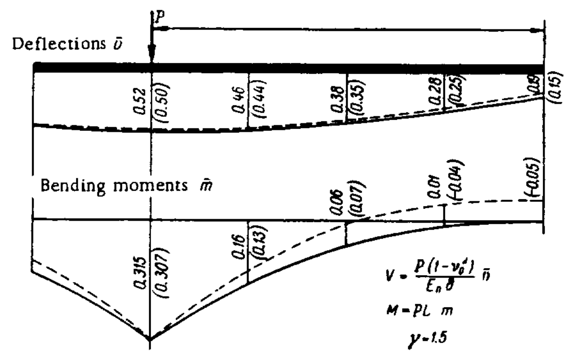


FIGURE 64.

This enables us to establish the values of λ at which transition from the finite-beam model to that of an infinitely long rigid beam occurs.

When a concentrated force acts, these values are as follows:

($\lambda < 0.86$ — rigid beam,

$0.86 < \lambda < 1.85$ — finite [elastic] beam,

$\lambda > 1.85$ — infinite beam.

The boundaries thus defined are identical with those obtained by the methods of the theory of elasticity, and in particular, by Gorbunov-Posadov's method.

§ 7. INFLUENCE OF LATERAL LOADING

1

The method explained above can be applied to many problems connected with beams on elastic foundations. Such a problem is, e.g., that of a load applied to the elastic foundation beyond the edges of a rigid beam.

* The comparison was performed for $\gamma = 1.5$, $H = \infty$, $\nu_0 = 0.3$.

Let a concentrated force G act on the elastic foundation at a distance a from the beam end (Figure 65). The following expressions are obtained for the vertical displacements of the surface of the elastic foundation in sections I, II, III, IV (Figure 65) (see sections 3 of Chapter I and 5 of Chapter II):

$$\left.\begin{aligned} V_I &= D_1 e^{\alpha(x+l)}, \\ V_{II} &= C_0 + \theta_0 x, \\ V_{III} &= D_2 e^{\alpha(l-x)} + D_3 e^{-\alpha(l-x)}, \\ V_{IV} &= D_4 e^{\alpha(l+a-x)}, \end{aligned}\right\} \tag{7.1}$$

where $\alpha = \sqrt{\frac{k}{2t}}$.

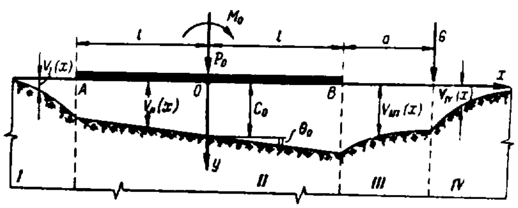


FIGURE 65.

The solution of this problem thus reduces to determining the six constants D_1, D_2, D_3, D_4, C_0 and θ_0.

Since the surface of the elastic foundation is assumed to be continuous, the following four independent conditions are obtained for the boundaries of parts I, II, III, and IV:

$$\left.\begin{aligned} &\text{at } x=-l && V_I = V_{II}, \\ &\text{at } x=l && V_{II} = V_{III}, \\ &\text{at } x=l+a && V_{III} = V_{IV}, \\ & && S_{III} - S_{IV} = G, \end{aligned}\right\} \tag{7.2}$$

where S_{III} and S_{IV} are the generalized shearing forces in parts III and IV respectively*.

With the aid of (7.1) and (5.3), conditions (7.2) can be written in the form:

$$\left.\begin{aligned} D_1 &= C_0 - \theta_0 l, \\ D_2 + D_3 &= C_0 + \theta_0 l, \\ D_2 e^{-\alpha a} + D_3 e^{\alpha a} - D_4 &= 0, \\ -D_2 e^{-\alpha a} + D_3 e^{\alpha a} + D_4 &= \frac{G}{2\alpha t}. \end{aligned}\right\} \tag{7.3}$$

* The last equation (7.2) states that $S(x)$ has a discontinuity of magnitude G at $x = l + a$. It is again assumed that $\psi(0) = 1$.

Solving (7.3) for D_1, D_2, D_3, D_4 we find:

$$\left.\begin{aligned} D_1 &= C_0 - \theta_0 l, \\ D_2 &= C_0 + \theta_0 l - \frac{G}{4\alpha l e^{\alpha a}}, \\ D_3 &= \frac{G}{4\alpha l e^{\alpha a}}, \\ D_4 &= (C_0 + \theta_0 l)\, e^{-\alpha a} + \frac{G}{4\alpha l} - \frac{G}{4\alpha l e^{2\alpha a}}. \end{aligned}\right\} \tag{7.4}$$

We determine C_0 and θ_0 from the equilibrium conditions of a beam acted upon by the external load and the reactions $q(x)$ and Q^{Φ} of the elastic foundation.

The distributed reactions $q(x)$ are, in accordance with (1.4) and (7.1):

$$q = k(C_0 + \theta_0 x). \tag{7.5}$$

The concentrated reactions Q^{Φ} are by (5.4):

$$\left.\begin{aligned} Q_A^{\Phi} &= S_I(-l) - S_{II}(-l), \\ Q_B^{\Phi} &= S_{II}(l) - S_{III}(l). \end{aligned}\right\} \tag{7.6}$$

The generalized shearing forces entering have to be obtained from (5.3) with the aid of (7.1) and (7.4); substitution of the expressions obtained in (7.6) yields:

$$\left.\begin{aligned} Q_A^{\Phi} &= 2t\,[\alpha C_0 - (1 + \alpha l)\,\theta_0], \\ Q_B^{\Phi} &= 2t\,[\alpha C_0 + (1 + \alpha l)\,\theta_0] - \frac{G}{e^{\alpha a}}. \end{aligned}\right\} \tag{7.7}$$

The equilibrium conditions of the beam, obtained by separately equating to zero the sum of the vertical projections of all forces and the sum of all moments about the beam center, yield:

$$\left.\begin{aligned} C_0 &= \frac{P_0 + \dfrac{G}{e^{\alpha a}}}{2(kl + 2\alpha t)}, \\ \theta_0 &= \frac{3\left(M_0 + \dfrac{G}{e^{\alpha a}}\, l\right)}{2l\,[kl^2 + 6t\,(1 + \alpha l)]}. \end{aligned}\right\} \tag{7.8}$$

Expressions (7.8) differ from (5.9) and (5.18) only by the presence of a term in the numerator containing G, through which allowance is made for the lateral load.

Analysis of (7.7) shows that this lateral load affects the concentrated reactions at the beam ends, thus altering the stress pattern. The influence of the load G decreases rapidly when the distance a is increased. When this load is distributed instead of being concentrated, allowance can be made for it in a similar way.

Consider two elastic beams of length $2l_1$ and $2l_2$ respectively, arranged in line and each acted upon by an external load. The displacements are shown in Figure 66.

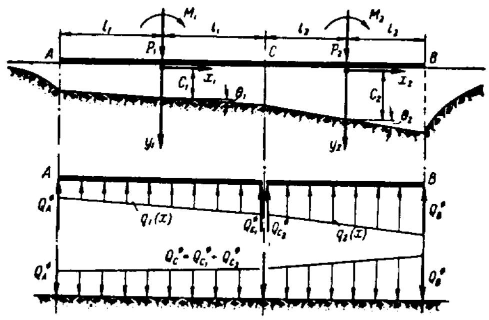


FIGURE 66.

Clockwise rotation will be taken as positive direction for the angles θ_1, θ_2 and external moments M_1 and M_2. For each beam a separate coordinate frame, with origin at the beam center, will be used.

The results of the previous example will be used to determine C_1, θ_1, C_2 and θ_2. The distributed reactions $q_1(x)$ and $q_2(x)$, acting on the first and second beam respectively, are:

$$\left.\begin{aligned} q_1(x) &= (C_1 + \theta_1 x) k, \\ q_2(x) &= (C_2 + \theta_2 x) k. \end{aligned}\right\} \tag{7.9}$$

The concentrated reactions Q_A^{ϕ} and Q_B^{ϕ} are:

$$\left.\begin{aligned} Q_A^{\phi} &= 2t\,[\alpha C_1 - (1 + \alpha l)\,\theta_1], \\ Q_B^{\phi} &= 2t\,[\alpha C_2 + (1 + \alpha l)\,\theta_2], \end{aligned}\right\} \tag{7.10}$$

where $\alpha = \sqrt{\frac{k}{2t}}$.

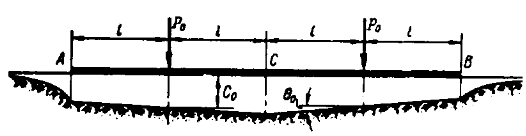


FIGURE 67.

The solution of this problem thus reduces to the determination of the six constants: C_1, θ_1, C_2, θ_2, $Q_{C_1}^{\phi}$ and $Q_{C_2}^{\phi}$. Therefore, a system of six algebraic equations has to be set up, which is represented in Table 5; the symbols used in it are identical with those appearing in Figure 66.

The first four equations in the table define the equilibrium conditions of each beam separately; the last two represent the continuity conditions of the vertical displacements of the foundation at the point where the two beams adjoin:

$$\text{at } x_1 = l_1 \ (x_2 = -l_2) : V_1 = V_2, \ S_1 = S_2.$$

The general solution is not given here, since it is much simpler to perform the calculations in each particular case by direct substitution of the numerical values in the equations.

Henceforth, only the case of two beams having equal lengths $2l$ and each acted upon at its center by a load P_0 will be considered (Figure 67).

From Table 5, equations (7.10), and the conditions of symmetry, we obtain:

$$\left.\begin{aligned} C_0 &= \frac{\left(\frac{kl^2}{3t} + \alpha l + 2\right)P_0}{2kl\left(\frac{kl^2}{3t} + \frac{4\alpha l}{3} + 2\frac{\alpha t}{kl} + 2\right)}, \\ \theta_0 &= \frac{\alpha P_0}{2kl\left(\frac{kl^2}{3t} + \frac{4\alpha l}{3} + 2\frac{\alpha t}{kl} + 2\right)}, \\ Q^{\Phi}_{C_1} = Q^{\Phi}_{C_2} &= \frac{\alpha l P_0}{kl\left(\frac{kl^2}{3t} + \frac{4\alpha l}{3} + 2\frac{\alpha t}{kl} + 2\right)}, \\ Q^{\Phi}_{A} = Q^{\Phi}_{B} &= \frac{\alpha t\left(\frac{kl^2}{3t} + 1\right)P_0}{kl\left(\frac{kl^2}{3t} + \frac{4\alpha l}{3} + 2\frac{\alpha t}{kl} + 2\right)}. \end{aligned}\right\} \quad (7.11)$$

TABLE 5.

No.	C_1	C_2	θ_1	θ_2	$Q^{\Phi}_{C_1}$	$Q^{\Phi}_{C_2}$	Right-hand part
1	$2(kl_1 + \alpha t)$	—	$-2t(1 + \alpha l_1)$	—	1	—	P_1
2	$-2\alpha t l_1$	—	$2l_1\left[(1 + \alpha l_1)t + \frac{kl_1^2}{3}\right]$	—	l_1	—	M_1
3	—	$2(kl_2 + \alpha t)$	—	$2t(1 + \alpha l_2)$	—	1	P_2
4	—	$2\alpha t l_2$	—	$2l_2\left[(1 + \alpha l_2)t + \frac{kl_2^2}{3}\right]$	—	$-l_2$	M_2
5	1	-1	l_1	l_2	—	—	0
6	—	—	$2t$	$-2t$	-1	-1	0

Let $\phi(y)$ be given by (5.22). The characteristics k, t, and α of the elastic foundation are then obtained from (5.23) and (5.24). We further assume that the elastic foundation is a semi-infinite plane ($H \to \infty$); in this case:

$$\left.\begin{aligned} k &= \frac{E_0\delta}{l(1-\nu_0^2)}\frac{\gamma}{2},\\ t &= \frac{E_0\delta l}{12(1-\nu_0)}\frac{3}{2\gamma},\\ \alpha &= \frac{\gamma}{l}\frac{\sqrt{2(1-\nu_0)}}{1-\nu_0}. \end{aligned}\right\} \tag{7.12}$$

Substitution of (7.12) in (7.11) yields:

$$\left.\begin{aligned} C_0 &= \frac{P_0(1-\nu_0^2)}{E_0\delta}\times\\ &\times\frac{\gamma^2+\frac{3}{4}\gamma\sqrt{2(1-\nu_0)}+\frac{3}{2}(1-\nu_0)}{\gamma\left[\gamma^2+\gamma\sqrt{2(1-\nu_0)}+\frac{3}{8\gamma}(1-\nu_0)\sqrt{2(1-\nu_0)}+\frac{3}{2}(1-\nu_0)\right]},\\ \theta_0 &= \frac{P_0(1-\nu_0^2)}{E_0\delta l}\times\\ &\times\frac{3\sqrt{2(1-\nu_0)}}{4\left[\gamma^2+\gamma\sqrt{2(1-\nu_0)}+\frac{3}{8\gamma}(1-\nu_0)\sqrt{2(1-\nu_0)}+\frac{3}{2}(1-\nu_0)\right]},\\ Q_{C_1}^{\phi} &= Q_{C_2}^{\phi} =\\ &=\frac{P_0\,3(1-\nu_0)\sqrt{2(1-\nu_0)}}{16\gamma\left[\gamma^2+\gamma\sqrt{2(1-\nu_0)}+\frac{3}{8\gamma}(1-\nu_0)\sqrt{2(1-\nu_0)}+\frac{3}{2}(1-\nu_0)\right]},\\ Q_A^{\phi} &= Q_B^{\phi} =\\ &=\frac{P_0\sqrt{2(1-\nu_0)}\left[\gamma^2+\frac{3}{4}(1-\nu_0)\right]}{4\gamma\left[\gamma^2+\gamma\sqrt{2(1-\nu_0)}+\frac{3}{8\gamma}(1-\nu_0)\sqrt{2(1-\nu_0)}+\frac{3}{2}(1-\nu_0)\right]}, \end{aligned}\right\} \tag{7.13}$$

and

$$q_1(x) = \frac{P_0}{2l}\,\frac{\gamma^2+\frac{3}{4}\gamma\sqrt{2(1-\nu_0)}+\frac{3}{2}(1-\nu_0)+\frac{3}{4}\gamma\sqrt{2(1-\nu_0)}\,\frac{x}{l}}{\gamma^2+\gamma\sqrt{2(1-\nu_0)}+\frac{3}{8\gamma}(1-\nu_0)\sqrt{2(1-\nu_0)}+\frac{3}{2}(1-\nu_0)}. \tag{7.14}$$

The dimensionless bending moments and shearing forces acting on the left beam, obtained from (7.13) and (7.14) for $\gamma = 1.5$, $\nu_0 = 0.3$, and a uniformly distributed load p are shown in Figure 68. The actual bending moments and shearing forces are respectively:

$$M = pl^2\bar{m}, \quad Q = pl\bar{Q}. \tag{7.15}$$

The same figure also shows the results obtained from (5.25) by neglecting the lateral load (broken lines, numbers in parentheses). It is seen that this additional load has a considerable influence, reducing the positive bending moments and the shearing forces.

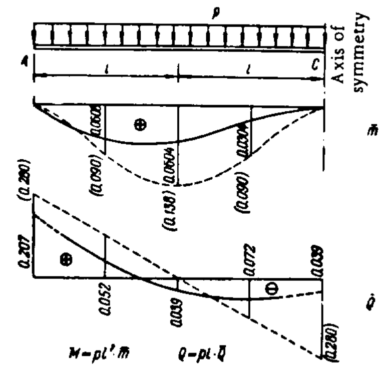


FIGURE 68.

3

The influence of a lateral load can be similarly treated in the case of elastic beams of finite length $2l$. Consider beams having dimensionless lengths $\lambda_1 = \frac{l_1}{L_1}$ and $\lambda_2 = \frac{l_2}{L_2}$, placed in line on an elastic foundation (Figure 69).

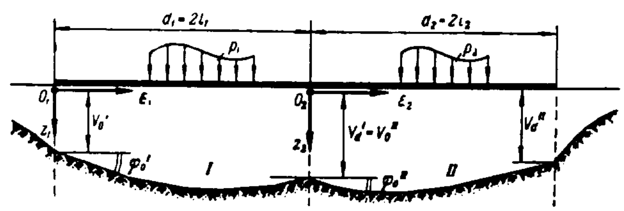


FIGURE 69.

The origin of coordinates for each beam is located at its left end. The boundary conditions are, in accordance with (6.7):

$$\left.\begin{array}{lll}
\text{at } \eta_1 = 0 & \text{a) } M_0^{\mathrm{I}} = 0 \quad \text{b) } N_0^{\mathrm{I}} = 2\alpha t V_0^{\mathrm{I}}; & \\
 & (\eta_2 = 0) & \\
\text{at } \eta_1 = \frac{2l_1}{L_1} & \text{c) } M_d^{\mathrm{I}} = M_0^{\mathrm{II}} = 0; \quad \text{d) } V_d^{\mathrm{I}} = V_0^{\mathrm{II}}; \quad \text{e) } N_d^{\mathrm{I}} = N_0^{\mathrm{II}}; & \\
\text{at } \eta_2 = \frac{2l_2}{L_2} & \text{f) } M_d^{\mathrm{II}} = 0; \quad \text{g) } N_d^{\mathrm{II}} = +\,2\alpha t V_d^{\mathrm{II}}, &
\end{array}\right\} \qquad (7.16)$$

where

$$L_1 = \sqrt[3]{\frac{2E_1J_1(1-\nu_0^2)}{E_0\delta}}, \quad L_2 = \sqrt[3]{\frac{2E_2J_2(1-\nu_0^2)}{E_0\delta}}.$$

The superscripts I and II indicate the corresponding beam.

The kinematic and statical magnitudes corresponding to the first beam are then determined except for the two parameters V_0^{I} and φ_0^{I}, as are those corresponding to the second beam, except for the three parameters V_0^{II}, φ_0^{II}, N_0^{II}.

Since V_0^{II} and N_0^{II} can be expressed through V_0^{I} and φ_0^{I} (cf. (7.16d, e and also, below, (7.17), (7.18)) the problem reduces to determining V_0^{I}, φ_0^{I}, φ_0^{II}. A system of three algebraic equations can be obtained from (7.16c, f, g), for the determination of these parameters.

This system is given in Table 6, where the following symbols are used:

$$K_V^{\mathrm{II}} = K_{NV}^{\mathrm{II}} - 2\alpha t K_{VV}^{\mathrm{II}},$$
$$K_N^{\mathrm{II}} = K_{NN}^{\mathrm{II}} - 2\alpha t K_{VN}^{\mathrm{II}},$$
$$K_\varphi^{\mathrm{II}} = K_{N\varphi}^{\mathrm{II}} - 2\alpha t K_{V\varphi}^{\mathrm{II}},$$

TABLE 6

No.	Boundary condition	V_0^{I}	φ_0^{I}	φ_0^{II}	Right-hand side
1	$M_{d_1}^{\mathrm{I}} = 0$	$K_{MV}^{\mathrm{I}}(d_1) + 2\alpha t K_{MN}^{\mathrm{I}}(d_1)$	$K_{M\varphi}^{\mathrm{I}}(d_1)$	—	$F_M^{\mathrm{I}}(d_1)$
2	$M_{d_2}^{\mathrm{II}} = 0$	$[K_{VV}^{\mathrm{I}}(d_2) + 2\alpha t K_{VM}^{\mathrm{I}}(d_2)] \times$ $\times K_{MV}^{\mathrm{II}}(d_2) + [K_{NV}^{\mathrm{I}}(d_2) +$ $+ 2\alpha t K_{NN}^{\mathrm{I}}(d_2)]\, K_{MN}^{\mathrm{II}}(d_2)$	$K_{V\varphi}^{\mathrm{I}}(d_1)\, K_{MV}^{\mathrm{II}}(d_2) +$ $+ K_{N\varphi}^{\mathrm{I}}(d_1)\, K_{MN}^{\mathrm{II}}(d_2)$	$K_{M\varphi}^{\mathrm{II}}(d_2)$	$F_V^{\mathrm{I}}(d_1)\, K_{MV}^{\mathrm{II}}(d_2) +$ $+ F_N^{\mathrm{I}}(d_1)\, K_{MN}^{\mathrm{II}}(d_2) +$ $+ F_M^{\mathrm{II}}(d_2)$
3	$N_{d_2}^{\mathrm{II}} = 2\alpha t V_{d_2}^{\mathrm{II}}$	$[K_{VV}^{\mathrm{I}}(d_1) + 2\alpha t K_{VN}^{\mathrm{I}}(d_1)] \times$ $\times K_V^{\mathrm{II}}(d_2) + [K_{NV}^{\mathrm{I}}(d_1) +$ $+ 2\alpha t K_{NN}^{\mathrm{I}}(d_1)]\, K_N^{\mathrm{II}}(d_2)$	$K_{V\varphi}^{\mathrm{I}}(d_1)\, K_V^{\mathrm{II}}(d_2) +$ $+ K_{N\varphi}^{\mathrm{I}}(d_1)\, K_N^{\mathrm{II}}(d_2)$	$K_\varphi^{\mathrm{II}}(d_2)$	$F_V^{\mathrm{I}}(d_1)\, K_V^{\mathrm{II}}(d_2) +$ $+ F_N^{\mathrm{I}}(d_1)\, K_N^{\mathrm{II}}(d_2) +$ $+ F_N^{\mathrm{II}} - 2\alpha t F_V^{\mathrm{II}}$

The general solution of this system is [cf. (6.4)]:
for the first beam

$$\left.\begin{aligned}
V^{\mathrm{I}}(\eta) &= (K_{VV}^{\mathrm{I}} + 2\alpha t K_{VN}^{\mathrm{I}})\, V_0^{\mathrm{I}} + K_{V\varphi}^{\mathrm{I}}\varphi_0^{\mathrm{I}} - F_V^{\mathrm{I}},\\
\varphi^{\mathrm{I}}(\eta) &= (K_{\varphi V}^{\mathrm{I}} + 2\alpha t K_{\varphi N}^{\mathrm{I}})\, V_0^{\mathrm{I}} + K_{\varphi\varphi}^{\mathrm{I}}\varphi_0^{\mathrm{I}} - F_\varphi^{\mathrm{I}},\\
M^{\mathrm{I}}(\eta) &= (K_{MV}^{\mathrm{I}} + 2\alpha t K_{MN}^{\mathrm{I}})\, V_0^{\mathrm{I}} + K_{M\varphi}^{\mathrm{I}}\varphi_0^{\mathrm{I}} - F_M^{\mathrm{I}},\\
N^{\mathrm{I}}(\eta) &= (K_{NV}^{\mathrm{I}} + 2\alpha t K_{NN}^{\mathrm{I}})\, V_0^{\mathrm{I}} + K_{N\varphi}^{\mathrm{I}}\varphi_0^{\mathrm{I}} - F_N^{\mathrm{I}};
\end{aligned}\right\} \tag{7.17}$$

for the second beam

$$\left.\begin{aligned} V^{\text{II}}(\eta) &= K_{VV}^{\text{II}}V_0^{\text{II}} + K_{V\varphi}^{\text{II}}\varphi_0^{\text{II}} + K_{VN}^{\text{II}}N_0^{\text{II}} - F_V^{\text{II}}, \\ \varphi^{\text{II}}(\eta) &= K_{\varphi V}^{\text{II}}V_0^{\text{II}} + K_{\varphi\varphi}^{\text{II}}\varphi_0^{\text{II}} + K_{\varphi N}^{\text{II}}N_0^{\text{II}} - F_\varphi^{\text{II}}, \\ M^{\text{II}}(\eta) &= K_{MV}^{\text{II}}V_0^{\text{II}} + K_{M\varphi}^{\text{II}}\varphi_0^{\text{II}} + K_{MN}^{\text{II}}N_0^{\text{II}} - F_M^{\text{II}}, \\ N^{\text{II}}(\eta) &= K_{NV}^{\text{II}}V_0^{\text{II}} + K_{N\varphi}^{\text{II}}\varphi_0^{\text{II}} + K_{NN}^{\text{II}}N_0^{\text{II}} - F^{\text{II}}. \end{aligned}\right\} \tag{7.18}$$

It can be shown that, as in the case of rigid beams, the lateral load reduces the concentrated reactions at the beam ends, thus considerably altering the stresses in the beam.

§ 8. BEAM ON AN ELASTIC DOUBLE-LAYER FOUNDATION

1

Consider a rigid beam of length $2l$ and width δ lying on a double-layer foundation and carrying an external load (Figure 70). Equations (5.1) and (5.15) of Chapter I are assumed to hold true for the elastic foundation; in other words, we are considering an elastic foundation with upper Winkler layer, whose properties are described by (5.16) and (5.18) of Chapter I.

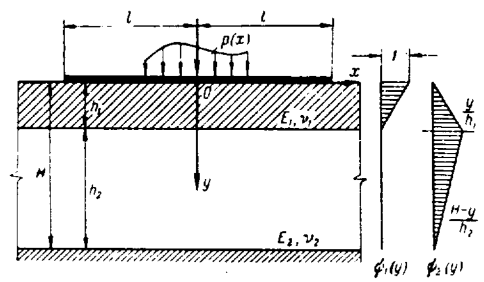


FIGURE 70.

In contrast to (5.2) of Chapter I, $\psi_2(y)$ is assumed to vary linearly over the entire height of the foundation (cf. Figure 70)*. From (5.4) and (5.15) of Chapter I, we obtain:

$$\left.\begin{aligned} k_1 &= K, & k_2 &= \frac{E_2\delta}{h_2(1-\nu_2^2)}, \\ t_1 &= 0, & t_2 &= \frac{E_2\delta h_2}{12(1+\nu_2)}. \end{aligned}\right\} \tag{8.1}$$

[cf. (5.7) of Chapter I]

Since the deflections of a rigid beam are equal to zero, the displacement of the surface of the elastic foundation beneath the beam (part III in Figure 71) is:

$$V_1 = C_0 + \theta_0 x, \tag{8.2}$$

* This derivation remains valid for any other function $\psi_2(y)$, since the latter determines only the coefficients k_2 and t_2.

where C_0 = vertical displacement of beam center, θ_0 = slope of beam.
From the first equation (5.16) of Chapter I and (8.2) we obtain:

$$-2t_2F'' + (K + k_2)F = C_0 + \theta_0 x. \tag{8.3}$$

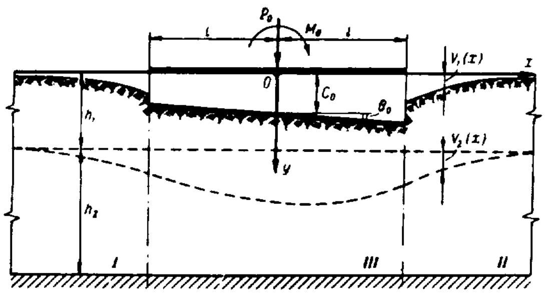


FIGURE 71.

The general integral of this equation is:

$$F(x) = C_1 \operatorname{ch} \beta x + C_2 \operatorname{sh} \beta x + \frac{C_0 + \theta_0 x}{K + k_2}, \tag{8.4}$$

where

$$\beta = \sqrt{\frac{K + k_2}{2t_2}}. \tag{8.5}$$

The constants of integration C_1 and C_2 are found from the conditions at the boundaries of the parts shown in Figure 71, which are:

at $x = -l$ $$V_2^{\mathrm{I}} = V_2^{\mathrm{III}}, \quad S_2^{\mathrm{I}} = S_2^{\mathrm{III}}; \tag{8.6}$$

at $x = l$ $$V_2^{\mathrm{III}} = V_2^{\mathrm{II}}, \quad S_2^{\mathrm{III}} = S_2^{\mathrm{II}}, \tag{8.7}$$

where the superscripts I, II, III indicate the corresponding part of the foundation.
Proceeding from (5.22) of Chapter I, we obtain:

$$\left.\begin{aligned} V_2^{\mathrm{I}} &= D_1 e^{\alpha_2(x+l)}, \\ V_2^{\mathrm{II}} &= D_2 e^{-\alpha_2(x-l)}, \end{aligned}\right\} \tag{8.8}$$

where

$$\alpha_2 = \sqrt{\frac{k_2}{2t_2}}.$$

By virtue of (5.17) of Chapter I, we obtain:

$$\left.\begin{aligned} S_2^{\mathrm{I}} &= 2\alpha_2 t_2 D_1 e^{\alpha_2(x+l)}, \\ S_2^{\mathrm{II}} &= -2\alpha_2 t_2 D_2 e^{-\alpha_2(x-l)}, \\ S_2^{\mathrm{III}} &= 2Kt_2\{C_1\beta \operatorname{sh} \beta x + C_2 \beta \operatorname{ch} \beta x\}. \end{aligned}\right| \tag{8.9}$$

Substitution of (8.4), (8.8), and (8.9) in the boundary conditions (8.6), (8.7) yields the following system of four algebraic equations by which the integration constants can be determined:

$$\left.\begin{aligned}
&-C_1\,\mathrm{ch}\,\beta l + C_2\,\mathrm{sh}\,\beta l + \frac{1}{K} D_1 = \frac{C_0 - \theta_0 l}{K + k_2},\\
&-C_1\,\mathrm{sh}\,\beta l + C_2\,\mathrm{ch}\,\beta l - \frac{\alpha_2}{\beta K} D_1 = 0,\\
&-C_1\,\mathrm{ch}\,\beta l - C_2\,\mathrm{sh}\,\beta l + \frac{1}{K} D_2 = \frac{C_0 + \theta_0 l}{K + k_2},\\
&-C_1\,\mathrm{sh}\,\beta l - C_2\,\mathrm{ch}\,\beta l + \frac{\alpha_2}{\beta K} D_2 = 0,
\end{aligned}\right\} \qquad (8.10)$$

By solving (8.10) we obtain C_1, C_2, D_1, and D_2 as functions of C_0 and θ_0. The vertical displacement C_0 of the beam center, and the slope θ, are found from the equilibrium conditions of the beam:

$$\sum Y = 0, \quad \sum M = 0.$$

We finally obtain:

$$\left.\begin{aligned}
C_1 &= -\frac{1}{lKk_2\,\mathrm{ch}\,\beta l\left[\lambda\dfrac{\mathrm{th}\,\beta l}{\beta l} + B_1\right]}\,\frac{P_0}{2},\\
C_2 &= -\frac{1}{l^2Kk_2\,\mathrm{ch}\,\beta l\left[\dfrac{\lambda}{\beta l}\left(1 - \dfrac{\mathrm{th}\,\beta l}{\beta l}\right) + \dfrac{B_2}{3}\right]}\,\frac{M_0}{2},\\
D_1 &= \frac{K}{K + k_2}\,\frac{(C_0 - \theta_0 l)\left[\mathrm{th}^2\beta l + 2\sqrt{1+\lambda}\,\mathrm{th}\,\beta l + 1\right] - \dfrac{C_0 + \theta_0 l}{\mathrm{ch}^2\beta l}}{2B_1B_3},\\
D_2 &= \frac{K}{K + k_2}\,\frac{(C_0 + \theta_0 l)\left[\mathrm{th}^2\beta l + 2\sqrt{1+\lambda}\,\mathrm{th}\,\beta l + 1\right] - \dfrac{C_0 - \theta_0 l}{\mathrm{ch}^2\beta l}}{2B_1B_3},\\
C_0 &= \frac{K + k_2}{lKk_2}\,\frac{B_1}{\lambda\dfrac{\mathrm{th}\,\beta l}{\beta l} + B_1}\,\frac{P_0}{2},\\
l\theta_0 &= \frac{K + k_2}{l^2Kk_2}\,\frac{B_2}{\dfrac{\lambda}{\beta l}\left(1 - \dfrac{\mathrm{th}\,\beta l}{\beta l}\right) + \dfrac{B_2}{3}}\,\frac{M_0}{2},
\end{aligned}\right\} \qquad (8.11)$$

where P_0 = sum of vertical loads, M_0 = sum of moments about beam center, considered as positive when acting clockwise, and:

$$\left.\begin{aligned}
B_1 &= 1 + \sqrt{1+\lambda}\,\mathrm{th}\,\beta l,\\
B_2 &= \sqrt{1+\lambda} + \mathrm{th}\,\beta l,\\
B_3 &= \frac{\sqrt{1+\lambda}}{1+\lambda}\,\mathrm{th}\,\beta l + 1,\\
\lambda &= \frac{K}{k_2}.
\end{aligned}\right\} \qquad (8.12)$$

From (8.11), the generalized displacements V_1, V_2, and the function F can be found by (5.16) of Chapter I, and (8.2) and (8.4) of this Chapter.

The reactions of the elastic foundation are obtained from (5.18) of Chapter I*, which, after substituting in it (8.4), yields:

$$q(x) = K\left[\frac{k_2}{K+k_2}(C_0 + \theta_0 x) - K(C_1 \operatorname{ch} \beta x + C_2 \operatorname{sh} \beta x)\right]. \tag{8.13}$$

The solution presented satisfies not only the equilibrium conditions of the beam and the condition of equal vertical displacements of beam and foundation, but also the statical boundary conditions: thus, the bending moments and shearing forces at simply supported beam ends, at which no loads act, are zero. This is due to the fact that the concentrated reactions Q^{Φ} existing in a single-layer foundation are absent in the double-layer model.

2

The case of symmetrical loading of a rigid beam will now be considered in more detail. Putting $M_0 = 0$ in (8.11) we obtain from (8.13):

$$q(x) = \frac{P_0}{2l} \frac{1 + \sqrt{1+\lambda} \operatorname{th} \beta l - \lambda \dfrac{\operatorname{ch} \beta x}{\operatorname{ch} \beta l}}{1 + \sqrt{1+\lambda} \operatorname{th} \beta l + \lambda \dfrac{\operatorname{th} \beta l}{\beta l}}, \tag{8.14}$$

where

$$\lambda = \frac{K}{k_2}, \quad \beta = \sqrt{\frac{K + k_2}{2t_2}}. \tag{8.15}$$

If $\psi_2(y)$ is linear (cf. Figure 70), the coefficients k_2 and t_2 are given by (8.1), the substitution of which in (8.15) finally yields:

$$q(x) = \frac{P_0}{2l} \bar{q}(x), \tag{8.16}$$

where

$$\bar{q}(x) = \frac{1 + \sqrt{1+\lambda} \operatorname{th}\left(\dfrac{l}{h_2}\sqrt{6\dfrac{1+\lambda}{1-\nu_0}}\right) - \lambda \dfrac{\operatorname{ch}\left(\dfrac{x}{h_2}\sqrt{6\dfrac{1+\lambda}{1-\nu_0}}\right)}{\operatorname{ch}\left(\dfrac{l}{h_2}\sqrt{6\dfrac{1+\lambda}{1-\nu_0}}\right)}}{1 + \sqrt{1+\lambda} \operatorname{th}\left(\dfrac{l}{h_2}\sqrt{6\dfrac{1+\lambda}{1-\nu_0}}\right) - \lambda \dfrac{\operatorname{th}\left(\dfrac{l}{h_2}\sqrt{6\dfrac{1+\lambda}{1-\nu_0}}\right)}{\dfrac{l}{h_2}\sqrt{6\dfrac{1+\lambda}{1-\nu_0}}}}. \tag{8.17}$$

Curves of the dimensionless function $\bar{q}(x)$ are given in Figure 72 for $\frac{l}{h_2} = 1$, $\nu_0 = 0$, and different values of $\lambda = \frac{K}{k_2}$.

It is seen that in all cases the reactions increase from the center toward the ends of the beam where they remain finite, varying directly with λ; when $\lambda = 0$ ($k_2 = \infty$) the double-layer foundation degenerates into an elastic

* When $q(x)$ is known, the bending moments and shearing forces of a rigid beam are obtained by the known methods of the strength of materials.

Winkler foundation. On the other hand, when $\lambda \to \infty$, the model becomes similar in its behavior to an elastic semi-infinite plane.

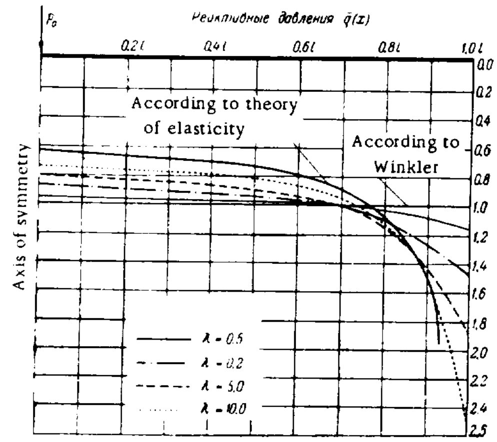


FIGURE 72.

3

When analyzing an elastic beam of finite length, the differential equation of the bending of the beam [cf. (1.1)],

$$EJV_1^{IV} = p - q \tag{8.18}$$

has to be taken into account together with (5.16) and (5.18) of Chapter I, which determine the deformations of the elastic double-layer foundation. Since it is assumed that the beam deflections are equal to the vertical displacements V_1 of the surface of the elastic foundation at the same points, the first equation (5.16) of Chapter I can be inserted into (8.18). From this, and from (5.18) of Chapter I, we then obtain:

$$\left.\begin{aligned} EJ\,[-2t_2F^{VI} + (K + k_2)\,F^{IV}] &= p - q, \\ -2Kt_2F'' + Kk_2F &= q. \end{aligned}\right\} \tag{8.19}$$

Elimination of $q(x)$ yields:

$$-EJ\frac{2t_2}{K}F^{VI} + EJ\frac{K+k_2}{K}F^{IV} - 2t_2F'' + k_2F = \frac{p(x)}{K}. \tag{8.20}$$

Equation (8.20) is a sixth-order ordinary differential equation with constant coefficients and can easily be integrated. The boundary conditions are given by (8.6) and (8.7). By including the statical conditions $M = 0$ and $Q = 0$, four independent equations can be established for each beam end; this number corresponds to the total number of integration constants*.

* The function $F(x)$ is determined for the entire beam length except for six constants which corresponds to the order of (8.20). Beyond the beam ends $F(x)$ is determined except for two constants (cf. (8.8)).

After $F(x)$ has been determined from (8.20) and the corresponding boundary conditions, the beam deflection and the reactions of the elastic foundation can be obtained from (5.16) and (5.18) of Chapter I; the bending moments and shearing forces are given by (1.11) and (1.12) of this chapter.

Chapter III

BENDING OF A RECTANGULAR PLATE ON AN ELASTIC SINGLE-LAYER FOUNDATION

§ 1. STATEMENT OF THE PROBLEM. DIFFERENTIAL EQUATIONS OF BENDING OF A PLATE ON A SINGLE-LAYER FOUNDATION

Consider a rectangular plate on an elastic foundation whose properties are described by (7.8) of Chapter I (Figure 73). The assumptions usually made in the theory of bending of thin plates will be deemed to apply to this case. Friction and adhesion between the plate and the surface of the elastic foundation will be neglected.

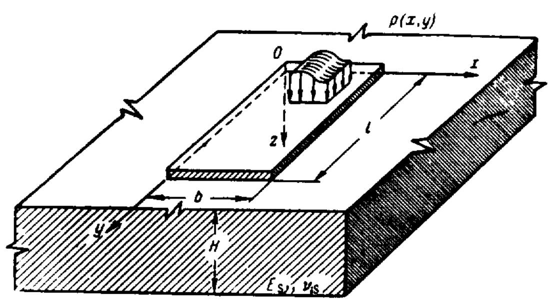


FIGURE 73.

The differential equation of bending of the plate, referred to cartesian coordinates, then becomes:

$$\nabla^2\nabla^2 w(x, y) = \frac{p^*}{D}. \tag{1.1}$$

(∇^2 denotes the Laplace operator) or, in expanded form:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p^*}{D}, \tag{1.2}$$

where $w = w(x, y)$ = vertical displacements of the plate surface, $p^* = p^*(x, y)$ = distributed load on the plate, $D = \frac{Eh^3}{12(1-\mu^2)}$ = flexural rigidity of the plate.

Although (1.2) is known as the equation of bending of thin plates, it can be applied to the analysis of most rectangular plates. It was shown by academician Galerkin /17/ that (1.2) is valid even when the ratio of the plate thickness to the smallest dimension in plan equals 1:3.

Since the plate lies on an elastic foundation, the distributed load consists of the given surface forces $p^*(x, y)$ and the reactions $q(x, y)$ of the elastic foundation:

$$p^*(x, y) = p(x, y) - q(x, y). \tag{1.3}$$

Since the reactions are unknown functions of the coordinates x, y, our problem consists in determining their distribution as well as the vertical displacements $w(x, y)$ of the plate. In addition, the equilibrium conditions of the plate and the condition of equal vertical displacements of the plate and the elastic-foundation surface have to be fulfilled.

It was established above (cf. (7.8), Chapter I) that the strains of a single-layer foundation under the action of a load distributed over its surface are given by the following differential equation:

$$-2t\nabla^2 w(x, y) + kw(x, y) = q(x, y)\,\psi(0), \tag{1.4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the Laplace operator.

$$k = \frac{E_0}{1-\nu_0^2}\int_0^H \psi'^2\,dz,$$

$$t = \frac{E_0}{4(1+\nu_0)}\int_0^H \psi^2\,dz$$

are the elastic characteristics of the single-layer foundation.

$\psi(z)$ = function of transverse distribution of displacements.

The deflections of the plate and the vertical displacements of the surface of the single-layer foundation are assumed to be equal. In addition, the load $q(x, y)$ acting on the foundation represents the reaction of the foundation on the plate. Hence, (1.2) and (1.4) have to be considered simultaneously.

Assume that $\psi(0) = 1$. Substitution of [(1.3) and] (1.4) in (1.1) yields:

$$\nabla^2\nabla^2 w - 2r^2\nabla^2 w + s^4 w = \frac{p}{D}, \tag{1.5}$$

where r^2 and s^4 are the generalized elastic characteristics of plate and foundation, defined as follows:

$$\left.\begin{aligned} r^2 &= \frac{E_0}{4(1+\nu_0)D}\int_0^H \psi^2(z)\,dz = \frac{t}{D},\\ s^4 &= \frac{E_0}{(1-\nu_0^2)D}\int_0^H \psi'^2(z)\,dz = \frac{k}{D},\end{aligned}\right\} \tag{1.6}$$

Also,

$$E_0 = \frac{E_s}{1-\nu_s^2}, \quad \nu_0 = \frac{\nu_s}{1-\nu_s}, \tag{1.7}$$

where E_s and ν_{s} are respectively the modulus of elasticity and Poisson's ratio for the material of the foundation (soil).

Differential equation (1.5) differs from that derived from the hypothesis of Winkler and Zimmermann by the additional term containing r^2, which makes allowance for the shearing stresses in the elastic foundation.

After $w(x, y)$ has been determined from (1.5) and the boundary conditions, the reactions $q(x, y)$ can be found from (1.4), the moments and shearing forces being given (Figure 74) by the known formulas of the theory of bending of plates:

$$\left.\begin{aligned} M_x &= -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right), \\ M_y &= -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right), \\ H = H_x &= -H_y = -D(1-\mu)\frac{\partial^2 w}{\partial x\,\partial y}, \\ N_x &= -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right), \\ N_y &= -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right). \end{aligned}\right| \tag{1.8}$$

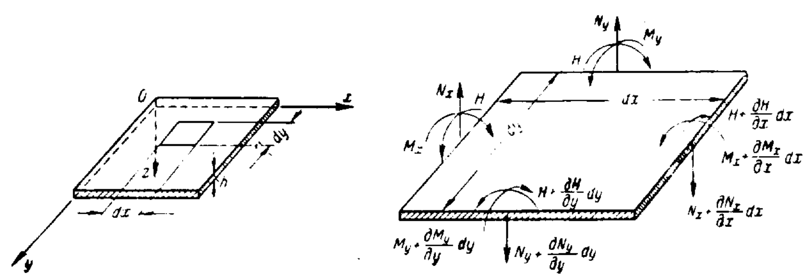


FIGURE 74.

Following Kirchhoff, the shearing forces N_x, N_y, and the torque H at the plate edges are usually replaced by the reduced shearing forces Q_x and Q_y which, for a rectangular plate, are:

$$\left.\begin{aligned} Q_x &= -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right], \\ Q_y &= -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2\,\partial y}\right]. \end{aligned}\right\} \tag{1.9}$$

§ 2. REDUCING THE PROBLEM OF THE BENDING OF A PLATE ON AN ELASTIC FOUNDATION TO ORDINARY DIFFERENTIAL EQUATIONS

1. General considerations

The problem of the bending of a plate on an elastic foundation can only be solved in closed form for a relatively small number of boundary conditions. In most cases the deflections $w(x, y)$ cannot be given as a finite polynomial in x and y.

Approximations, based on series expansions, are therefore mostly used to solve problems concerning the bending of plates. One example is the solution by single or double trigonometric series (the problems of Maurice Lévy and Navier respectively).

Although simple and convenient for practical calculations, this method is only applicable to certain particular boundary conditions. The more general method of reduction to ordinary differential equations will accordingly be used here.

The general variational method of reduction to ordinary differential equations, as applied to the problem of the bending of a rectangular plate, is discussed thoroughly in "Structural Mechanics of Thin-Walled Three-Dimensional Systems", by V. Z. Vlasov. Only the application of this method to the analysis of thin plates on elastic foundations will be dealt with here. No restrictions are imposed on the boundary conditions at the longitudinal and lateral edges of the plate, whose thickness may vary exponentially in one or both directions.

2. Reducing the two-dimensional problem to a one-dimensional problem

We shall consider the x-axis to lie in the lateral, and the y-axis, in the longitudinal direction of the plate (Figure 75).

The unknown deflections of the plate $w(x, y)$ will be represented as a finite series:

$$w(x, y) = \sum_{k=1}^{n} W_k(y)\chi_k(x). \tag{2.1}$$

in which the dimensionless functions $\chi_k(x)$ determine the lateral distribution of the deflection of the plate, and are assumed to be known. The unknown functions $W_k(y)$, which have the dimension of length, can, in accordance with their physical meaning, be called generalized deflections.

Different expressions may be chosen for the functions $\chi_k(x)$, provided they are linearly independent and satisfy the boundary conditions at the longitudinal edges of the plate. The simplest example of such a system, satisfying the boundary conditions:

$$\chi_k(0) = \chi_k(b) = 0.$$

is a series in $\sin\frac{k\pi x}{b}$, where k is an integer and assumes all values between 1 and n.

The introduction of the finite expansion (2.1) is equivalent to reducing the plate to a system having a finite number of degrees of freedom in the lateral direction and an infinite number of degrees of freedom in the longitudinal direction. It is also equivalent to reducing the two-dimensional problem of the theory of elasticity to a one-dimensional problem, since the deflections $w(x, y)$ are obtained by determining the n functions of the single variable $W_k(y)$.

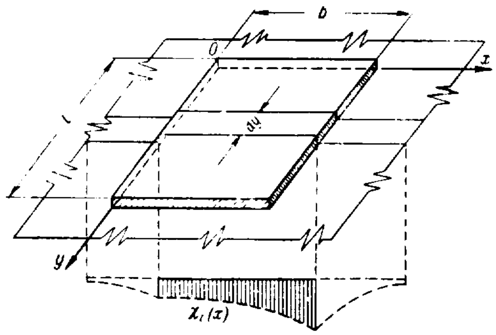


FIGURE 75.

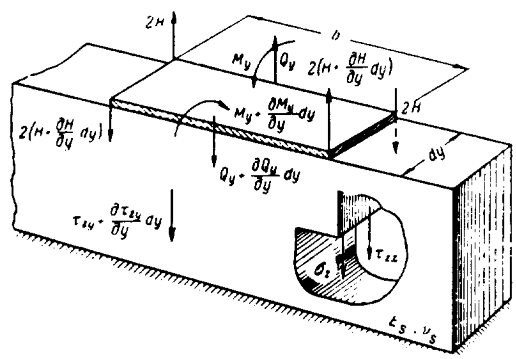


FIGURE 76.

To determine the unknown functions $W_k(y)$, we consider the equilibrium of an elementary slab (composed of elements of plate and foundation) bounded by the cross sections $y = \text{const}$ and $y + dy = \text{const}$ (Figures 75 and 76). In accordance with the principle of virtual displacements, the equilibrium conditions will be expressed by equating to zero the total work done by all external and internal forces acting on this slab over any virtual displacement.

Let the i-th virtual displacement of the plate element be a cylindrical bending in the vertical plane. The deflections of the upper surface are determined by one of the functions $\chi_i(x)$, while the corresponding generalized deflection $W_i(y) = 1$. Since all the virtual displacements of the plate are defined by the n linearly independent functions $\chi_i(x)$, ($i = 1 \ldots n$), n independent conditions of equilibrium can be established, from which the n unknown functions $W_k(y)$ ($k = 1 \ldots n$) may be determined.

3. Generalized equilibrium conditions of the elementary slab

The elementary slab defined above consists of a compressible layer belonging to the elastic foundation and of an element lying on it (Figure 76).

To establish the generalized conditions of equilibrium, we consider separately the forces acting on the respective elements of the plate and of the elastic foundation.

In addition to the given external load, moments M_y, $M_y + \frac{\partial M_y}{\partial y} dy$, and forces Q_y, $Q_y + \frac{\partial Q_y}{\partial y} dy$, due to the remainder of the plate, act on the sections $y = \text{const}$ and $y + dy = \text{const}$ respectively of the plate element. At the corners, concentrated vertical forces $2H$ and $2\left(H + \frac{\partial H}{\partial y} dy\right)$ act, which, according to Kirchhoff, result from replacing the torques by statically equivalent additional shearing forces.

All these forces, with the positive directions shown in Figure 76, are external forces relative to the plate element. The internal forces are due to the stresses in the longitudinal sections $x = \text{const}$; these stresses can be reduced to bending moments M_x and reduced shearing forces Q_x.

The external forces acting on the element of the elastic foundation are the normal and shearing stresses acting on the vertical edges $y = \text{const}$ and $y + dy = \text{const}$. The internal forces are due to the normal stresses σ_x, σ_z, and the shearing stresses τ_{zx} and τ_{xz}.

In order to establish the generalized equilibrium conditions for the system considered, the work done by each of these forces separately will now be calculated.

The work of the internal forces acting on the plate element is equal to the work done by the bending moments M_x and shearing forces Q_x in the corresponding deformations of the element. On the strength of the assumption usually made in the theory of bending of plates, the work done by the shearing forces Q_x will be equal to zero; the work done by the bending moments is:

$$\int M_x \chi_i'' \, dx. \tag{2.2}$$

where the integral is taken from 0 to b.

The work done by the external forces acting on the plate element consists of:

a) Work done by the given external load

$$G_i = \int p(x, y) \chi_i(x) \, dx, \tag{2.3}$$

where the integral contains not only the distributed load $p(x, y)$, but also the concentrated shearing forces and moments, and is understood as a Stieltjes integral. We can therefore rewrite (2.3) as follows:

$$G_i = \int p(x, y) \chi_i(x) \, dx + \sum p_c(y) \chi_i(c) + \sum m_c(y) \chi_i'(c), \tag{2.4}$$

where $p_c(y)$ and $m_c(y)$ are the concentrated shearing loads and moments respectively, acting along the lines $x = x_c$ which include the reduced shearing forces $Q_x(0)$, $Q_x(b)$ and the bending moments $M_x(0)$, $M_x(b)$.

b) Work done by the reduced shearing forces

$$Q_y \text{ and } Q_y + \frac{\partial Q_y}{\partial y}\,dy:$$

$$\int \frac{\partial Q_y}{\partial y}\chi_i(x)\,dx. \qquad (2.5)$$

c) Work done by the concentrated shearing forces

$$2H \text{ and } 2\left(H + \frac{\partial H}{\partial y}\,dy\right)$$

$$-2\left[\frac{\partial H}{\partial y}\chi_i\right]^*. \qquad (2.6)$$

Henceforth, the brackets with an asterisk will denote the difference between the values of the magnitude inside the brackets at $x = 0$ and $x = b$ respectively.

The work done by the bending moments M_y over the virtual displacements $\chi_i(x)$ ($W_i(y) = 1$) is zero.

In accordance with the assumption made for a single-layer foundation,

$$u(x, y, z) = 0, \quad v(x, y, z) = 0,$$

the work done by the internal and external forces acting on the element of the elastic foundation is represented by the work done by the normal stresses σ_z and the shearing stresses τ_{zx}, τ_{zy} (at y = const), and $\tau_{zy} + \frac{\partial \tau_{zy}}{\partial y}\,dy$ (at $y + dy$ = const) in compressive and shearing deformations respectively. This work will be denoted by $R_i(y)$.

The integral equilibrium condition of the slab element is thus:

$$\int M_x \chi_i''\,dx + \int \frac{\partial Q_y}{\partial y}\chi_i\,dx - 2\left[\frac{\partial H}{\partial y}\chi_i\right]^* + R_i + G_i = 0 \qquad (2.7)$$

$$(i = 1, 2, 3, \ldots, n).$$

4. Work done by the external and internal forces acting on the elastic foundation

Consider first the most important case of free longitudinal plate edges.

From the condition of continuity of the vertical displacements $w(x, y)$ over the surface of the entire elastic foundation, we obtain for the region beyond the plate edges:

$$w(x, y) = \sum_{k=1}^{n} W_k(y)\,\chi_{0k}(x). \qquad (2.8)$$

where $W_k(y)$ are the generalized vertical displacements, and the dimensionless functions $\chi_{0k}(x)$ are:

$$\left.\begin{aligned} &\text{at } x \leqslant 0 && \chi_{0k} = \chi_k(0)\,e^{\alpha x}, \\ &\text{at } x \geqslant b && \chi_{0k} = \chi_k(b)\,e^{-\alpha(x-b)}, \end{aligned}\right\} \qquad (2.9)$$

where $\alpha = \sqrt{\frac{k}{2t}}$, k and t = generalized characteristics of the elastic foundation (cf. (1.4)).

This means that the element cut from the elastic foundation behaves exactly like the plane single-layer model considered earlier: the vertical displacements of the surface vary exponentially. Each virtual displacement

$$w_i(x, y) = 1 \cdot \chi_i(x)$$

corresponds to a uniquely defined displacement of the surface of the elastic foundation beyond the plate edges.

The displacements of an elastic single-layer foundation are determined by (7.2) of Chapter I. Therefore, the virtual displacements of the surface of the elastic foundation

$$\bar{w}_i(x, y) = 1 \cdot \chi_i(x)$$

correspond to the virtual displacements inside the elastic foundations:

$$w_i(x, y, z) = 1 \cdot \chi(x) \psi(z). \tag{2.10}$$

The work done by the shearing stresses τ_{zy} and $\tau_{zy} + \frac{\partial \tau_{zy}}{\partial y} dy$, distributed along the edges $y = \text{const}$ and $y + dy = \text{const}$, over the virtual displacements given by (2.10) is:

$$\int_{-\infty}^{+\infty} dx \int_0^H \frac{\partial \tau_{zy}}{\partial y} \chi_i(x) \psi(z)\, dz. \tag{2.11}$$

The work done by the internal stresses σ_z and τ_{zx} in the deformations corresponding to the virtual displacements (2.10) is respectively:

$$\left.\begin{aligned} &-\int_{-\infty}^{+\infty} dx \int_0^H \sigma_z \chi_i(x) \psi'(z)\, dz, \\ &-\int_{-\infty}^{+\infty} dx \int_0^H \tau_{zx} \chi_i'(x) \psi(z)\, dz. \end{aligned}\right\} \tag{2.12}$$

where, in accordance with (6.4), (6.5), (7.1), and (7.2) of Chapter I and (2.1) of this chapter,

$$\left.\begin{aligned} \sigma_z &= \frac{E_0}{1-\nu_0^2} \psi'(z) \sum_{k=1}^{n} W_k(y) \chi_k(x), \\ \tau_{zy} &= \frac{E_0}{2(1+\nu_0)} \psi(z) \sum_{k=1}^{n} W_k'(y) \chi_k(x), \\ \tau_{zx} &= \frac{E_0}{2(1+\nu_0)} \psi(z) \sum_{k=1}^{n} W_k(y) \chi_k'(x). \end{aligned}\right\} \tag{2.13}$$

Only the stresses in the foundation directly below the plate can be calculated by (2.13). In order to determine the normal and shearing stresses beyond the boundaries of the plate, the functions $\chi_k(x)$ and $\chi_k'(x)$ in (2.13) have to be replaced by $\chi_{0k}(x)$ and $\chi_{0k}'(x)$ respectively.

Substitution of (2.13) in (2.11) and (2.12) yields, after integration:

$$R_i(y) = \sum_{k=1}^{n} \left\{2t \int \chi_k \chi_i \, dx + \frac{t}{\alpha} [[\chi_k \chi_i]] \right\} W_k'' -$$
$$- \sum_{k=1}^{n} \left\{k \int \chi_k \chi_i \, dx + 2t \int \chi_k' \chi_i' \, dx + 2\alpha t [[\chi_k \chi_i]] \right\} W_k, \tag{2.14}$$

where

$$\left.\begin{aligned} k &= \frac{E_0}{1-\nu_0^2} \int_0^H \psi'^2(z)\, dz, \\ t &= \frac{E_0}{4(1+\nu_0)} \int_0^H \psi^2(z)\, dz, \\ \alpha &= \sqrt{\frac{k}{2t}}. \end{aligned}\right\} \tag{2.15}$$

The integrals in (2.14) are taken from 0 to b. The double brackets denote the sum of the values, at $x = 0$ and $x = b$, of the magnitude inside the brackets.

When the longitudinal edges of the plate are built-in or simply supported $(\chi_k(0) = \chi_k(b) = 0)$, and the elastic foundation is not strained beyond the plate boundaries, (2.14) reduces to:

$$R_i(y) = \sum_{k=1}^{n} \left\{2t \int \chi_k \chi_i \, dx \, W_k'' - \left(k \int \chi_k \chi_i \, dx + 2t \int \chi_k' \chi_i' \, dx\right) W_k\right\}. \tag{2.16}$$

A particular case of (2.14) is obtained when one plate edge is free and the other built-in.

5. Second method for obtaining the generalized equilibrium conditions

We shall show now how the strains of the elastic foundation beneath the plate can be taken into account in a different way.

As before it will be assumed that the deflections of the plate are given by (2.1). The generalized deflections $W_k(y)$ will be determined from the equilibrium conditions for a plate element bounded by the planes $y = \text{const}$, $y + dy = \text{const}$ (Figure 77). It will be assumed that the plate element is acted upon by reactions of the elastic foundation, in addition to the external load and to the forces transmitted from the remainder of the plate.

Let the i-th virtual displacement of the plate element be a cylindrical bending, the deflections of the element being determined by the functions $\chi_i(x)$ $(W_i(y) = 1)$; the generalized equilibrium conditions of the plate element are then, [cf. (2.7)]:

$$\int M_x \chi_i'' \, dx + \int \frac{\partial Q_y}{\partial y} \chi_i \, dx - 2\left[\frac{\partial H}{\partial y} \chi_i\right]^* + R_i + G_i = 0,$$

where $G_i = G_i(y)$ = work done by external load, and $R_i = R_i(y)$ = work done by reactions of the elastic foundation over the virtual displacements:

$$\bar{w}_i(x, y) = 1 \cdot \chi_i(x).$$

The reactions of the elastic foundation consist, as in the case of a beam in a state of plane strain, of the distributed reactions $q(x, y)$ and the fictitious forces $Q^{\Phi}(y)$* acting at the plate edges (Figure 77).

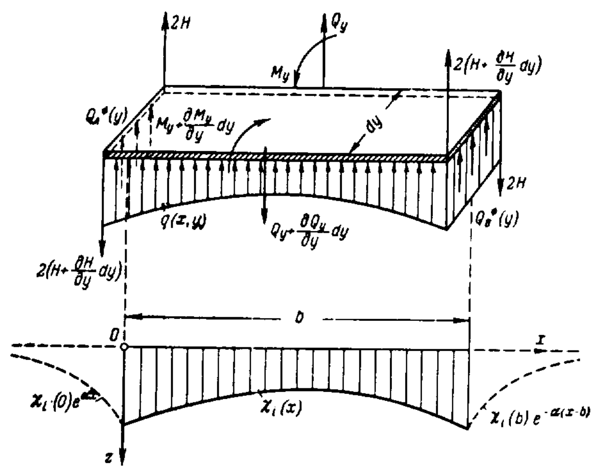


FIGURE 77.

We obtain from (2.1) and (1.4):

$$q(x, y) = k\sum_{k=1}^{n} W_k\chi_k - 2t\sum_{k=1}^{n} W_k\chi_k'' - 2t\sum_{k=1}^{n} W_k''\chi_k, \tag{2.17}$$

where k and t are given by (2.15).

The work done by the distributed reactions (2.17) over the displacement χ_i is then:

$$\begin{aligned} -\int q(x, y)\chi_i\,dx = 2\sum_{k=1}^{n} t\int \chi_k\chi_i\,dx\,W_k'' - \\ -\sum_{k=1}^{n}\left(k\int \chi_k\chi_i\,dx + 2t\int \chi_k'\chi_i'\,dx - 2t\,[\chi_k'\chi_i]^*\right)W_k. \end{aligned} \tag{2.18}$$

By the concentrated forces Q^{Φ} allowance is made for the influence of the free foundation beyond the plate edges. In other words, these forces result from the work done by all the forces acting on the element of the foundation over the virtual displacement χ_{0i} of the foundation beyond the plate edges.

* $Q^{\Phi}(y)$ are given as forces per unit length.

We thus obtain:

$$\left.\begin{aligned} Q_A^\Phi &= -2t\sum_{k=1}^{n}\left\{W_k[\chi_k'(0)-\alpha\chi_k(0)]+\frac{1}{2\alpha}\chi_k(0)W_k^{\cdot\cdot}\right\},\\ Q_B^\Phi &= 2t\sum_{k=1}^{n}\left\{W_k[\chi_k'(b)+\alpha\chi_k(b)]-\frac{1}{2\alpha}\chi_k(b)W_k^{\cdot\cdot}\right\}.\end{aligned}\right\} \tag{2.19}$$

The work done by these reactions over the virtual edge displacements $\chi_i(0)$ and $\chi_i(b)$ respectively is:

$$-[Q_A^\Phi\chi_i(0)+Q_B^\Phi\chi_i(b)] = -\sum_{k=1}^{n}\left\{(2t[\chi_k'\chi_i]^*+2\alpha t[[\chi_k\chi_i]])W_k-\frac{t}{\alpha}[[\chi_k\chi_i]]W_k^{\cdot\cdot}\right\}. \tag{2.20}$$

Finally adding together (2.18) and (2.20), we obtain:

$$\begin{aligned} R_i(y) = \sum_{k=1}^{n}&\left\{2t\int\chi_k\chi_i\,dx+\frac{t}{\alpha}[[\chi_k\chi_i]]\right\}W_k^{\cdot\cdot}-\\ &-\left\{k\int\chi_k\chi_i\,dx+2t\int\chi_k'\chi_i'\,dx+2\alpha t\,[[\chi_k\chi_i]]\right\}W_k.\end{aligned} \tag{2.21}$$

which coincides with (2.14).

6. Solution of the ordinary differential equation

The forces and moments M_x, H, Q_y and their derivatives entering in (2.7) are by (1.8), (1.9), and (2.1):

$$\left.\begin{aligned} M_x &= -D\sum_{k=1}^{n}(\mu W_k^{\cdot\cdot}\chi_k+W_k\chi_k^{\cdot\cdot}),\\ H &= -D\sum_{k=1}^{n}(1-\mu)W_k^{\cdot}\chi_k',\\ \frac{\partial H}{\partial y} &= -D\sum_{k=1}^{n}(1-\mu)W_k^{\cdot\cdot}\chi_k',\\ Q_y &= -D\sum_{k=1}^{n}\{W_k^{\cdot\cdot\cdot}\chi_k+(2-\mu)W_k^{\cdot}\chi_k^{\cdot\cdot}\},\\ \frac{\partial Q_y}{\partial y} &= -D\sum_{k=1}^{n}\{W_k^{IV}\chi_k+(2-\mu)W_k^{\cdot\cdot}\chi_k^{\cdot\cdot}\}.\end{aligned}\right\} \tag{2.22}$$

Substitution of (2.22) in (2.7) yields:

$$\begin{aligned} &\sum_{k=1}^{n}W_k^{IV}\int\chi_k\chi_i\,dx+\sum_{k=1}^{n}W_k^{\cdot\cdot}(2-\mu)\int\chi_k^{\cdot\cdot}\chi_i\,dx-\\ &\quad-2\sum_{k=1}^{n}W_k^{\cdot\cdot}(1-\mu)[\chi_k'\chi_i]^*+\sum_{k=1}^{n}W_k^{\cdot\cdot}\mu\int\chi_k\chi_i^{\cdot\cdot}dx+\\ &\quad\quad+\sum_{k=1}^{n}W_k\int\chi_k^{\cdot\cdot}\chi_i^{\cdot\cdot}dx-\frac{R_i}{D}-\frac{G_i}{D}=0.\end{aligned} \tag{2.23}$$

Integrating by parts:

$$\left.\begin{aligned}\int \chi_k'' \chi_i\, dx &= [\chi_k' \chi_i]^* - \int \chi_k' \chi_i'\, dx,\\ \int \chi_k \chi_i''\, dx &= [\chi_k \chi_i']^* - \int \chi_k' \chi_i'\, dx,\end{aligned}\right\} \tag{2.24}$$

and inserting (2.21), equation (2.23) becomes:

$$\sum_{k=1}^{n} a_{ik} W_k^{\mathrm{IV}} - 2\sum_{k=1}^{n} (b_{ik} + \rho_{ik}^0)\, W_k'' + \sum_{k=1}^{n} (c_{ik} + s_{ik}^0)\, W_k - G_i = 0 \tag{2.25}$$

$$(i = 1, 2, 3, \ldots, n);$$

where

$$\left.\begin{aligned}
a_{ik} &= \sum D \int \chi_k \chi_i\, dx,\\
b_{ik} &= \sum D \left\{ \int \chi_k' \chi_i'\, dx - \frac{\mu}{2} [\chi_k \chi_i' + \chi_k' \chi_i]^* \right\},\\
c_{ik} &= \sum D \int \chi_k'' \chi_i''\, dx,\\
\rho_{ik}^0 &= t \int \chi_k \chi_i\, dx + \frac{t}{2a} [[\chi_k \chi_i]],\\
s_{ik}^0 &= k \left\{ \int \chi_k \chi_i\, dx + \frac{2t}{k} \int \chi_k' \chi_i'\, dx + \frac{2at}{k} [[\chi_k \chi_i]] \right\}
\end{aligned}\right\} \tag{2.26}$$

Here $D = \frac{Eh^3}{12(1-\mu^2)}$ = flexural rigidity of plate; μ = Poisson's ratio of plate material; $E_0 = \frac{E_s}{1-\nu_s^2}$, $\nu_0 = \frac{\nu_s}{1-\nu_s}$ = elastic constants of foundation material.

Expressions (2.26) are applicable to a plate whose thickness varies stepwise in the x direction. The integrals are calculated for each part whose flexural rigidity D is uniform; the expression in brackets with asterisk then denotes the difference between the values of $(\chi_k \chi_i' + \chi_k' \chi_i)$ at the ends of each part. The summation is extended over all such parts.

The coefficients in (2.25) evidently depend only on the functions $\chi_h(x)$. These coefficients are symmetrical:

$$a_{ik} = a_{ki}, \quad b_{ik} = b_{ki}, \quad c_{ik} = c_{ki}, \quad \rho_{ik}^0 = \rho_{ki}^0, \quad s_{ik}^0 = s_{ki}^0, \tag{2.27}$$

in accordance with the reciprocity theorem of Maxwell and Betti.

The free term in (2.25) represents the functions $G_i(y)$, obtained from (2.4) as the generalized load per unit length corresponding to the virtual displacements $\chi_i(x)$.

By letting i assume successively different integral values between 1 and n, we obtain a system of n ordinary differential equations with constant coefficients for the determination of the n unknown functions W_k. By virtue of (2.27), this system has a symmetrical structure. All the equations will be of the fourth order in each unknown function.

7. Determining the moments and shearing forces

Solving the system (2.25) for given boundary conditions at $x=0$ and $x=b$ we obtain all the functions $W_k(y)$; the function $w(x, y)$ can then be determined by (2.1).

We can then rewrite (1.8) and (1.9) as follows:

$$
\left.\begin{aligned}
M_x &= -D\sum_{k=1}^{n}(\mu W''_k\chi_k + W_k\chi''_k), && \text{(a)}\\
M_y &= -D\sum_{k=1}^{n}(W''_k\chi_k + \mu W_k\chi''_k), && \text{(b)}\\
H = H_x = -H_y &= -D\sum_{k=1}^{n}(1-\mu)\,W'_k\chi'_k, && \text{(c)}\\
N_x &= -D\sum_{k=1}^{n}(W''_k\chi_k + W_k\chi'''_k), && \text{(d)}\\
N_y &= -D\sum_{k=1}^{n}(W'''_k\chi_k + W'_k\chi''_k), && \text{(e)}\\
Q_x &= -D\sum_{k=1}^{n}\{(2-\mu)\,W''_k\chi'_k + W_k\chi'''_k\}, && \text{(f)}\\
Q_y &= -D\sum_{k=1}^{n}\{W'''_k\chi_k + (2-\mu)\,W'_k\chi''_k\}. && \text{(g)}
\end{aligned}\right\} \qquad (2.28)
$$

§3. GENERALIZED INTERNAL FORCES. BOUNDARY CONDITIONS AT THE LATERAL PLATE EDGES

As already mentioned above, the functions $W_i(y)$ represent the generalized plate deflections corresponding to the virtual displacements $\chi_i(x)$. The derivative of the generalized deflection therefore defines the generalized slope $\varphi_i(y)$. The geometrical magnitudes $W_i(y)$ and $\varphi_i(y)$ correspond to the generalized bending moments $M_i(y)$ and the generalized shearing forces $N_i(y)$, exactly as in the theory of the bending of beams.

The generalized bending moments M_i represent the work done by all the bending moments M_y acting in the section $y=\text{const}$ over the corresponding virtual rotations $\frac{\partial w_i}{\partial y} = \varphi_i\chi_i$ $(\varphi_i = 1)$; the generalized shearing forces N_i represent the work done by the shearing forces N_y, the twisting moments H, and the shearing stresses τ_{zy} acting in the section $y=\text{const}$ over the virtual displacements $\bar{w}_i = W_i\chi_i$ $(W_i = 1)$.

The shearing stresses τ_{zy}, acting in the section $y=\text{const}$ of the elastic foundation, are, by virtue of (6.5), (7.2) of Chapter I, and (2.1) of this chapter:

$$\tau_{zy} = \frac{E_0}{2(1+\nu_0)}\frac{\partial w}{\partial y}\psi(z) = \frac{E_0}{2(1+\nu_0)}\psi(z)\sum_{k=1}^{n}W'_k(y)\,\chi_k(x). \qquad (3.1)$$

The work of these stresses over any virtual displacement of the elastic foundation must be calculated over the entire cross section $y=\text{const}$, i.e., for $-\infty \leqslant x \leqslant +\infty$ and $0 \leqslant z \leqslant H$.

The i-th virtual displacement of an arbitrary point $M(x, y, z)$ of a single-layer foundation is:

$$w_i(x, y, z) = W_i(y)\chi_i(x)\phi(z). \tag{3.2}$$

With $W_i(y) = 1$ the virtual work done by the shearing stresses τ_{zy} is therefore :

$$\iint \tau_{zy}\chi_i(x)\phi(z)\,dx\,dz. \tag{3.3}$$

Substituting (2.9), (2.15) and (3.1) in (3.3), and integrating, we obtain the following expression for the virtual work done by the shearing stresses acting in the section $y = \text{const}$:

$$\sum_{k=1}^{n}\left\{2t\int\chi_k\chi_i\,dx + \frac{t}{\alpha}\,[[\chi_k\chi_i]]\right\}W'_k. \tag{3.4}$$

[The integration limits are in fact: $x = 0$ and $x = b$]

By (2.1), [(2.24)], (2.28 b, c, e,), and (3.4), the generalized moments and shearing forces acting in the section $y = \text{const}$ are:

$$M_i = -\sum_{k=1}^{n}\left\{\left(\sum D\int\chi_k\chi_i\,dx\right)W''_k - \mu\sum D\left(\int\chi'_k\chi'_i\,dx - [\chi'_k\chi_i]^*\right)W_k\right\} \tag{3.5}$$

$$N_i = -\sum_{k=1}^{n}\left(\sum D\int\chi_k\chi_i\,dx\right)W'''_k + \sum_{k=1}^{n}\left\{(2-\mu)\sum D\left(\int\chi'_k\chi'_i\,dx - [\chi'_k\chi_i]^*\right) + 2t\int\chi_k\chi_i\,dx + \frac{t}{\alpha}\left[[\chi_k\chi_i]\right]\right\}W'_k \tag{3.6}$$

$$(i = 1,\ 2,\ \ldots,\ n),$$

[where the second term on the right includes the virtual work done by the force $\frac{\partial H}{\partial x}$].

The boundary conditions for $y = 0$ and $y = b$ can now be expressed in integral form with the aid of (3.5) and (3.6).

Since (2.25) is a system of order $4n$, the functions W_k are determined except for $4n$ constants. It is therefore necessary to add $4n$ boundary conditions to (2.25) in order to obtain a complete solution. It is seen from (2.1), (3.5), and (3.6) that $2n$ boundary conditions can be specified at each edge $y = 0$, $y = l$. If the plate is built-in, the boundary conditions are given as generalized displacements; when the edges are free the boundary conditions are given as generalized forces, and when the edges are simply supported, the boundary conditions are given partly as forces and partly as displacements.

§ 4. SELECTING THE FUNCTION OF THE LATERAL DISTRIBUTION OF THE DEFLECTIONS. BOUNDARY CONDITIONS AT THE LONGITUDINAL PLATE EDGES

The functions $\chi_k(x)$, which determine the lateral distribution of the displacements of the plate, can be selected in different ways provided they satisfy the geometrical boundary conditions at the longitudinal plate edges and are linearly independent. Several methods for selecting the functions χ_k, and the corresponding properties of the matrix of (2.25), will now be considered.

1. Eigenfunctions of the transverse vibrations of a beam

The eigenfunctions of the transverse vibrations of a beam having uniform cross section can be selected as functions $\chi_k(x)$, when the boundary conditions for the beam are similar to those at the longitudinal plate edges.

We begin with a short discussion of the theory of eigenfunctions*. The free vibrations of a single-span massive beam of length b are described by the differential equation:

$$X^{\mathrm{IV}} = \frac{\mu^4}{b^4} X, \tag{4.1}$$

where $X = X(x)$ is the deflection of the beam axis at x, and μ = parameter characterizing frequency of natural vibrations of beam.

The general integral of (4.1) is:

$$X(x) = C_1 \sin\frac{\mu x}{b} + C_2 \cos\frac{\mu x}{b} + C_3 \operatorname{sh}\frac{\mu x}{b} + C_4 \operatorname{ch}\frac{\mu x}{b}. \tag{4.2}$$

The constants C_1, C_2, C_3, C_4 and the parameter μ are determined from the boundary conditions at the beam ends $x = 0$ and $x = b$. The form of the function $X(x)$ depends therefore on these conditions.

Some particular boundary conditions will be considered.

1. Both beam ends simply supported.

In this case the boundary conditions for $X(x)$ are:

$$\left.\begin{array}{ll} \text{at } x = 0 & X(0) = X''(0) = 0, \\ \text{at } x = b & X(b) = X''(b) = 0. \end{array}\right\} \tag{4.3}$$

Substitution of (4.2) yields:

$$\left.\begin{array}{r} C_2 + C_4 = 0, \\ -C_2 + C_4 = 0, \\ C_1 \sin\mu + C_2 \cos\mu + C_3 \operatorname{sh}\mu + C_4 \operatorname{ch}\mu = 0, \\ -C_1 \sin\mu - C_2 \cos\mu + C_3 \operatorname{sh}\mu + C_4 \operatorname{ch}\mu = 0. \end{array}\right\} \tag{4.4}$$

* For a more thorough treatment of this problem, see Vlasov, V. Z. "Structural Mechanics of Thin-Walled Three-Dimensional Systems", see also section 2 of Chapter VI.

The first two equations give $C_2 = C_4 = 0$. The remaining two equations then reduce to:

$$\left.\begin{aligned} C_1 \sin\mu + C_3 \operatorname{sh}\mu = 0, \\ -C_1 \sin\mu + C_3 \operatorname{sh}\mu = 0. \end{aligned}\right\} \tag{4.5}$$

Since, for a nontrivial solution, all constants cannot vanish simultaneously, the determinant of (4.5) must be equal to zero. The following transcendental equation is then obtained for μ:

$$\sin\mu = 0,$$

which has an infinite number of real roots $\mu_m (m = 1, 2, 3, \ldots)$ equal to:

$$\pi,\ 2\pi,\ 3\pi,\ \ldots,\ m\pi. \tag{4.6}$$

In accordance with (4.6) we obtain a complete system of eigenfunctions:

$$X_m(x) = \sin\frac{m\pi x}{b} \qquad (m = 1, 2, 3, \ldots),$$

which determine an infinite number of modes of the natural vibrations.

2. Both beam ends built-in. The boundary conditions are in this case:

$$\begin{aligned} &\text{at } x = 0 \qquad && X(0) = X'(0) = 0, \\ &\text{at } x = b \qquad && X(b) = X'(b) = 0. \end{aligned} \tag{4.7}$$

Substitution of (4.2) yields, as before, a system of algebraic equations in the integration constants C_1, C_2, C_3, C_4. Equating the determinant of this system to zero, we obtain:

$$\cos\mu \operatorname{ch}\mu = 1,$$

The roots of this equation are:

$$0;\ 4.730;\ 7.853;\ 10.996;\ \ldots;\ \frac{2m+1}{2}\pi. \tag{4.8}$$

The eigenfunctions $X_m(x)$ determined by (4.8) are in this case:

$$X_m(x) = \sin\frac{\mu_m x}{b} - \operatorname{sh}\frac{\mu_m x}{b} - \alpha_m\left(\cos\frac{\mu_m x}{b} - \operatorname{ch}\frac{\mu_m x}{b}\right).$$

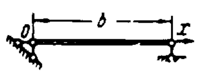
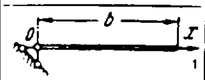
where

$$\alpha = \frac{\sin\mu_m - \operatorname{sh}\mu_m}{\cos\mu_m - \operatorname{ch}\mu_m}.$$

No further cases will be discussed. Proceeding from (4.2) and the specified boundary conditions of the problem, the eigenfunctions $X_m(x)$ can be determined for continuous multi-span beams. In each case a system of homogeneous equations will be obtained for the determination of the integration constants. By equating to zero the determinant of this system we obtain a transcendental equation for the parameter μ, whose roots define, together with the boundary conditions, all the eigenfunctions corresponding to the problem considered.

Table 7 contains the eigenfunctions of $\xi = \frac{x}{b}$ for the six basic cases of boundary conditions of a single-span beam, together with the corresponding transcendental equations and some of their roots.

TABLE 7

	Beam supports	Boundary conditions at $x=0$	Boundary conditions at $x=l$	Eigen-function $X(\xi)$	Coefficient α	Transcendental equation	Roots of transcendental equation μ_1	μ_2	μ_3	μ_4	general formula for μ_n $(n>4)$
1		$X=0$ $X''=0$	$X=0$ $X''=0$	$\sin\mu\xi$	—	$\sin\mu=0$	$\pi=$ $=3.1416$	2π	3π	4π	$n\pi$
2		$X=0$ $X'=0$	$X=0$ $X'=0$	$\sin\mu\xi-$ $-\operatorname{sh}\mu\xi-$ $-\alpha(\cos\mu\xi-$ $-\operatorname{ch}\mu\xi)$	$\frac{\sin\mu-\operatorname{sh}\mu}{\cos\mu-\operatorname{ch}\mu}$	$\cos\mu\operatorname{ch}\mu=$ $=1$	4.7300	7.8532	10.9956	14.1372	$\frac{2n+1}{2}\pi$
3		$X''=0$ $X'''=0$	$X''=0$ $X'''=0$	$\sin\mu\xi+$ $+\operatorname{sh}\mu\xi-$ $-\alpha(\cos\mu\xi+$ $+\operatorname{ch}\mu\xi)$	$\frac{\sin\mu-\operatorname{sh}\mu}{\cos\mu-\operatorname{ch}\mu}$	$\cos\mu\operatorname{ch}\mu=$ $=1$	4.7300	7.8532	10.9956	14.1372	$\frac{2n+1}{2}\pi$
4		$X=0$ $X'=0$	$X''=0$ $X'''=0$	$\sin\mu\xi-$ $-\operatorname{sh}\mu\xi-$ $-\alpha(\cos\mu\xi-$ $-\operatorname{ch}\mu\xi)$	$\frac{\sin\mu+\operatorname{sh}\mu}{\cos\mu+\operatorname{ch}\mu}$	$\cos\mu\operatorname{ch}\mu=$ $=-1$	1.8751	4.6941	7.8548	10.9955	$\frac{2n-1}{2}\pi$
5		$X=0$ $X''=0$	$X=0$ $X'=0$	$\sin\mu\xi-$ $-\alpha\operatorname{sh}\mu\xi$	$\frac{\sin\mu}{\operatorname{sh}\mu}$	$\operatorname{tg}\mu=$ $=\operatorname{th}\mu$	3.9266	7.0685	10.2102	13.3520	$\frac{4n+1}{4}\pi$
6		$X=0$ $X''=0$	$X''=0$ $X'''=0$	$\sin\mu\xi+$ $+\alpha\operatorname{sh}\mu\xi$	$\frac{\sin\mu}{\operatorname{sh}\mu}$	$\operatorname{tg}\mu=$ $=\operatorname{th}\mu$	3.9266	7.0685	10.2102	13.3520	$\frac{4n+1}{4}\pi$

In order to simplify the use of the eigenfunctions, values of $X_m(x)$ and of their first three derivatives, multiplied by constant factors:

$$\frac{b}{\mu_m}X_m'(b),\ \frac{b^2}{\mu_m^2}X_m''(x),\ \frac{b^3}{\mu_m^3}X_m'''(x),$$

are given in Tables 5 to 10 of the appendix for nine intermediate sections and the two end sections $x=0$ and $x=b$ of the beam.

The eigenfunctions determined in this way possess some properties which have very important practical applications. Firstly, they are orthogonal over the entire length of the beam:

$$\int_0^b X_m(x)\,X_n(x)\,dx=0. \qquad (m\neq n)$$

The second derivatives of the eigenfunctions are also orthogonal:

$$\int_0^b X_m''(x)\, X_n''(x)\, dx = 0. \qquad (m \neq n)$$

It then follows from (4.1) that all even derivatives of these functions are orthogonal.

The corresponding integrals for $m=n$ are different from zero and independent of the boundary conditions at $x=0$; they can be expressed through the function and its derivatives at $x=b$ only:

$$\int_0^b X_m^2(x)\, dx = \frac{b}{4}\left[X_m^2 - 2X_m'X_m''' + (X_m'')^2\right]_{x=b}.$$

Returning to the problem of a plate on an elastic foundation, we take as eigenfunctions $\chi_k(x)$ the functions of the lateral distribution of the deflections $X_k(x)$.

The solution of (2.25) is simplified when $\chi_k = X_k$. Thus, for the symmetrical problem (cases 1, 2, and 3 in Table 7), the system (2.25) can be divided into two independent sub-systems, each containing only even or only odd terms of (2.1). The system (2.25) cannot be divided when schemes 4, 5, or 6 in Table 7 apply.

Furthermore, by virtue of the orthogonality of the eigenfunctions and their second derivatives, the coefficients a_{ik} and c_{ik} vanish for $i \neq k$.

2. Trigonometric functions

The problem considered is solved most easily when the longitudinal plate edges are simply supported (case 1 in Table 7). In this case the eigenfunctions degenerate into the trigonometric functions $\sin\frac{k\pi x}{b}$, all derivatives of which are orthogonal. Hence, since $\chi_k = \sin\frac{k\pi x}{b} = 0$ at $x=0$ and $x=b$, all coefficients (2.26) vanish for $i \neq k$. System (2.25) then reduces to separate independent equations of the fourth order in $W_k(y)$.

Trigonometric functions may also be used when the longitudinal plate edges are free, or when one edge is simply supported while the other is free (cases 3 and 6 in Table 7). The elastic line can in this case be approximated by a series consisting of the first (linear) terms of the eigenfunctions and of the trigonometric functions $\sin\frac{k\pi x}{b}$.

The function $w(x, y)$ for a plate with free ends (case 3) is then:
for symmetrical loading

$$w(x, y) = W_0(y)\, 1 + W_1(y)\sin\frac{\pi x}{b} + W_3(y)\sin\frac{3\pi x}{b} + \ldots; \qquad (4.9)$$

for antisymmetrical loading

$$w(x, y) = W_0(y)\left(1 - \frac{2x}{b}\right) + W_2(y)\sin\frac{2\pi x}{b} + W_4(y)\sin\frac{4\pi x}{b} + \ldots \qquad (4.10)$$

For a plate simply supported at one of its edges (case 6) we obtain:

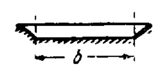
$$w(x,\ y) = W_0(y)\frac{x}{b} + W_1(y)\sin\frac{\pi x}{b} + W_2(y)\sin\frac{2\pi x}{b} + \ldots \qquad (4.11)$$

When the functions $\chi_k(x)$ are defined by (4.9) or (4.10), and $D = \text{const}$ system (2.25) can be represented in the form of Tables 8 and 9 for symmetrical and antisymmetrical loading respectively.

TABLE 8

Matrix of ordinary differential equations for symmetrical loading

$$w_{xy} = W_0 \cdot 1 + \sum_{k=1}^{n} W_k \sin\frac{k\pi x}{b} \quad \text{при } k = 1, 3, 5, \ldots, n$$

i	W_0	W_1	W_3	...	W_n	Free term	Displacement
0	$a_{00}D^4 - 2\rho^0_{00}D^2 + s^0_{00}$	$a_{01}D^4 - 2(b_{00} + \rho^0_{01})D^2 + s^0_{01}$	$a_{03}D^4 - 2(b_{03} + \rho^0_{03})D^2 + s^0_{03}$	...	$a_{0n}D^4 - 2(b_{0n} + \rho^0_{0n})D^2 + s^0_{0n}$	G_0	$\chi_0 = 1$ 
1	•	$a_{11}D^4 - 2(b_{11} + \rho^0_{11})D^2 + (s^0_{11} + c_{11})$	0	...	0	G_1	$\chi_1 = \sin\frac{\pi x}{b}$ 
3	•	0	$a_{33}D^4 - 2(b_{33} + \rho^0_{33})D^2 + (s^0_{33} + c_{33})$	...	0	G_3	$\chi_3 = \sin\frac{3\pi x}{b}$ 
...	....	....	....	...	...	...	....
n	•	0	0	...	$a_{nn}D^4 - 2(b_{nn} + \rho^0_{nn})D^2 + (s^0_{nn} + c_{nn})$	G_n	$\chi_n = \sin\frac{n\pi x}{b}$

The only nonzero terms in these matrices are those of the principal diagonal, the first row, and the first column.

The symbols D^4 and D^2 in these tables denote respectively the fourth and second derivative of the function indicated at the head of the column. The coefficients:

$$a_{00},\ \rho^0_{00},\ s^0_{00},\ a_{01},\ \ldots,\ c_{nn},\ s^0_{nn}$$

are obtained from (2.26), by substituting in them the expressions:

for a symmetrical load,

$$\chi_0 = 1,\ \chi_n = \sin\frac{n\pi x}{b} \qquad (\text{at } n = 1, 3, 5, 7, \ldots);$$

for an antisymmetrical load,

$$\chi_0 = 1 - \frac{2x}{b},\ \chi_m = \sin\frac{m\pi x}{b} \qquad (\text{at } m = 2, 4, 6, \ldots).$$

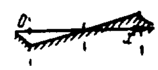
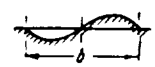
The free terms in the last column but one of each table represent the work done by the external load over the corresponding displacement $W_i\chi_i$ when $W_i = 1$.

These examples show that in order to obtain a complete solution it is necessary to solve an infinite number of ordinary differential equations.

TABLE 9

Matrix of ordinary differential equations for antisymmetrical loading

$$w(x, y) = W_0\left(1 - \frac{2x}{b}\right) + \sum_{k=2}^{m} W_k \sin\frac{k\pi x}{b} \quad \text{при } k = 2, 4, 6, \dots, m$$

i	W_0	W_2	W_4	...	W_m	Free term	Displacements
0	$a_{00}D^4 - 2(b_{00} + p^0_{00})D^2 + s_{00}$	$a_{02}D^4 - 2(b_{02} + p^0_{02})D^2 + s^0_{02}$	$a_{04}D^4 - 2(b_{04} + p^0_{04})D^2 + s^0_{04}$	...	$a_{0m}D^4 - 2(b_{0m} + p^0_{0m})D^2 + s^0_{0m}$	G_0	$\chi_0 = 1 - \frac{2x}{b}$ 
2	•	$a_{22}D^4 - 2(b_{22} + p^0_{22})D^2 + (c_{22} + s^0_{22})$	0	...	0	G_2	$\chi_2 = \sin\frac{2\pi x}{b}$ 
4	•	•	$a_{44}D^4 - 2(b_{44} + p^0_{44})D^2 + (c_{44} + s^0_{44})$	...	0	G_4	$\chi_4 = \sin\frac{4\pi x}{b}$ 
...	...	...	...	...	...	...	...
	•	•	•	...	$a_{mm}D^4 - 2(b_{mm} + p_{mm})D^2 + (c_{mm} + s_{mm})$	G_n	$\chi_m = \sin\frac{m\pi x}{b}$ при $m = 2, 4, 6\dots$

Since, however, the series representing trigonometric functions or eigenfunctions converge rapidly, it suffices in practice to take a small number of terms in (2.1). Thus, if the load distribution is nearly uniform in the x-direction, two or three terms in (4.9) and (4.10) are sufficient in order to obtain satisfactory accuracy. This is also true for the other methods of plate support.

When only a limited number of terms are taken in (2.1), the bending moments M_x and shearing forces N_x and Q_x can be determined directly from the equilibrium conditions instead of from (2.28).

3. Fulfilling the statical [equilibrium] conditions at the longitudinal edges

As already stated, the functions $\chi_k(x)$ are selected in order to satisfy the geometrical boundary conditions at the longitudinal plate edges. The fulfilment of the statical [equilibrium] conditions depends on the type of the boundary conditions and the form of the functions $\chi_k(x)$, and is, as a rule, only approximate; this does not, however, introduce large errors in the calculations.

The statical [equilibrium] conditions at $x=0$ and $x=b$ are particular cases of the conditions of equilibrium which, generally speaking, must be fulfilled at all points. On the other hand, the equilibrium conditions were included only in an integral form when establishing (2.25). In this solution, as usual for variational methods, the average deviation from the exact solution and from the strict fulfillment of the equilibrium conditions is small; at certain points, however, and particularly at the boundaries $x=0$, $x=b$; the equilibrium conditions may not be satisfied.

The following examples will make this point clear.

a) In the case of free plate end $x=0$, functions $X_k(x)$, can be selected as eigenfunctions which satisfy the following boundary conditions:

at $x=0$ $$X''(0)=X'''=0,\ X(0)\neq 0,\ X'(0)\neq 0.$$

The following expressions are then obtained from (2.28a) and (2.28f):

$$\left.\begin{aligned} M_x(0) &= -D\sum_{k=1}^{n}\mu W_k\chi_k,\\ Q_x(0) &= -D\sum_{k=1}^{n}(2-\mu)W_k\chi_k'. \end{aligned}\right\} \tag{4.12}$$

These equations are identical with the statical boundary conditions only for certain values of M_x and Q_x; in particular, they are not identical with the homogeneous boundary conditions:

$$M_x(0)=0,\ Q_x(0)=0 \tag{4.13}$$

Hence, the statical boundary conditions at the free ends are only approximately satisfied in the general case, the accuracy depending on the number of terms taken in the series expansions.

b) When the plate is supported on hinges at $x=0$, the eigenfunctions must satisfy the condition:

$$\chi(0)=0,\ \chi''(0)=0.$$

It then follows from (2.28a) that:

$$M_x(0)=0. \tag{4.14}$$

The homogeneous boundary condition (4.14) is thus identically satisfied in this case. If an external moment is applied at $x=0$, the resulting non-homogeneous boundary condition will not be satisfied, irrespective of the number of terms taken in the expansion. This contradiction is, however, purely formal, since in a section an infinitesimal distance from the boundary section we shall obtain a value for M_x which is very close to the actual value by taking a sufficient number of terms in the series expansion.

It will be shown in section 5 how the statical [boundary] conditions at the longitudinal edges can be approximated with a minimum number of terms, by means of a different selection or an extension of the system of functions $\chi_k(x)$.

§ 5. SELECTING THE FUNCTIONS OF THE LATERAL DISTRIBUTION OF THE DEFLECTIONS BY THE STATICAL-EQUILIBRIUM METHOD

Eigenfunctions or trigonometric functions are not the only possible choice for the functions $\chi_k(x)$. They can also be obtained by means of the statical-equilibrium method.

We consider the plate element of width dy as an ordinary beam, its elastic line being determined by the boundary conditions. Different elastic lines can be obtained by varying the point of application of a concentrated force acting on this beam; these lines, which are third-order curves, are then taken as functions $\chi_k(x)$ (Figure 78).

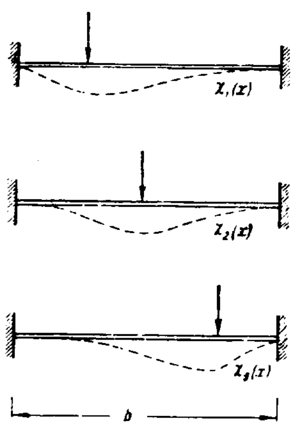


FIGURE 78.

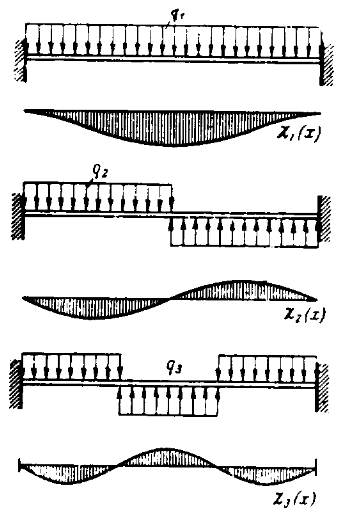


FIGURE 79.

In the same way we can apply a distributed load to the beam. By assuming different laws of variation with x of this load, we can obtain different forms of the functions χ_k from the differential equation:

$$\chi_k^{\mathrm{IV}} = \frac{p}{EJ}$$

and the boundary conditions. A certain function χ_k will correspond to each type of loading. With a uniformly distributed load, of differing intensity in different parts of the beam (positive or negative) (Figure 79), the deflections χ_k of each part are represented by fourth-order parabolas when the rigidity EJ of each part is uniform. Since the functions $\chi_k(x)$ and their derivatives may be expressed differently in different parts of the beam, we shall consider the integrals on the right sides of (2.26) as the sum of the integrals taken over all these parts.

This method is more general than the method of eigenfunctions. This follows from the following property of the eigenfunctions:

$$\chi^{\mathrm{IV}}(x) = k\chi(x).$$

In the particular case when the load varies like the ordinates of the elastic line, the eigenfunctions themselves will represent the elastic line.

Further, when a shearing force $Q(x)$ acts at the free end $x = 0$, or when a moment M_x is applied at a free or hinged end, the elastic lines of the beam, due to a force $Q_x(0) = 1$ and moment $M_x(0) = 1$ respectively, can be included in functions $\chi_k(x)$. A better approximation to the exact solution at, and near, the free end $x = 0$ is thus obtained than by taking a finite number of terms in the expansions of the eigenfunctions.

This method is also simpler than the method of eigenfunctions, used when analyzing complex structures, such as continuous plates and plates of variable thickness whose rigidities vary exponentially in the x direction.

An elementary strip of width dy of such a structure can, depending on the cross section of this structure, be considered either as a stepped or as a continuous beam. By applying an external load to such a beam, we obtain the functions $\chi_k(x)$ by the known methods of the theory of structures.

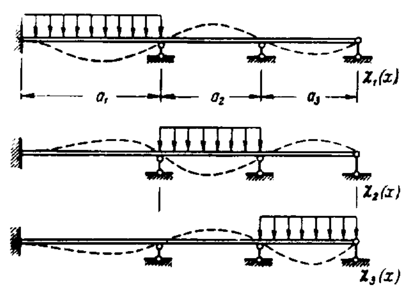


FIGURE 80.

Different functions $\chi_k(x)$, approximating the deflections $w(x)$ for $y = \text{const}$, are obtained by varying the external load. Figure 80 shows the functions $\chi_k(x)$ obtained as deflections of a continuous beam under the action of three types of loads, approximating the deflections of a continuous plate built-in along the edge $x = 0$ and having rigid supports parallel to the y axis at $x = a_1$, $x = a_1 + a_2$, and $x = a_1 + a_2 + a_3$.

§ 6. PLATE SIMPLY SUPPORTED AT OPPOSITE ENDS

1

Consider a rectangular plate on an elastic foundation, simply supported along the longitudinal edges (Figure 81).

The functions of the lateral distribution of the deflections are assumed to be:

$$\chi_n(x) = \sin\frac{n\pi x}{b}. \tag{6.1}$$

Because of the orthogonality of these functions and their derivatives, we find that all coefficients (2.26) having different subscripts vanish, while those with equal subscripts become:

$$\left.\begin{aligned} a_{nn} = D\frac{b}{2}, \quad b_{nn} = D\frac{n^2\pi^2}{2b}, \quad c_{nn} = D\frac{n^4\pi^4}{2b^3}, \\ \rho^0_{nn} = t\frac{b}{2}, \quad s^0_{nn} = k\frac{b}{2}\left[1 + \frac{2t}{k}\frac{n^2\pi^2}{b^2}\right] \end{aligned}\right\} \qquad (6.2)$$

System (2.25) thus reduces to n independent equations:

$$a_{nn}W_n^{\mathrm{IV}} - 2(b_{nn} + \rho^0_{nn})W''_n + (c_{nn} + s^0_{nn})W_n = G_n, \qquad (6.3)$$

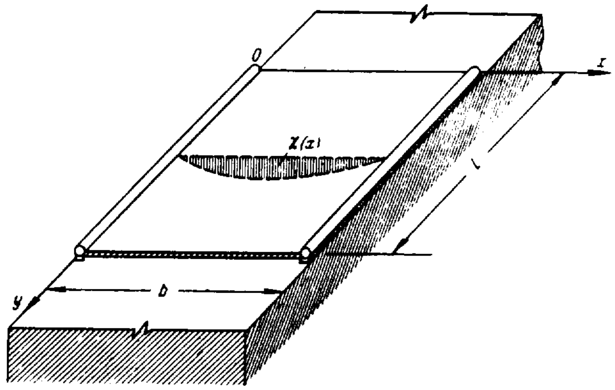


FIGURE 81.

The subscript n will henceforth be omitted. It should be kept in mind that the coefficients (6.2) and the function W correspond to a certain value of n in (2.1).

2

It is convenient to write (6.3) in dimensionless coordinates. Introducing a new variable $\eta = \frac{y}{l}$ and noting that:

$$\frac{dW}{dy} = \frac{1}{l}\frac{dW}{d\eta}, \qquad \frac{d^2W}{dy^2} = \frac{1}{l^2}\frac{d^2W}{d\eta^2} \text{ etc.}$$

we can rewrite (6.3) in the form:

$$W^{\mathrm{IV}} - 2r^2W'' + s^4W = \frac{l^4}{a}G. \qquad (6.4)$$

where

$$\left.\begin{aligned} r^2 &= \frac{b + \rho^0}{a}l^2, \\ s^4 &= \frac{c + s^0}{a}l^4. \end{aligned}\right\} \qquad (6.5)$$

Substitution of (6.2) in (6.5) yields:

$$\left.\begin{aligned} r^2 &= n^2\pi^2\frac{l^2}{b^2}+\frac{1}{2\alpha^2L^2}\left(\frac{l}{L}\right)^2,\\ s^4 &= n^4\pi^4\frac{l^4}{b^4}+\left[1-\frac{n^2\pi^2}{\alpha^2b^2}\right]\left(\frac{l}{L}\right)^4, \end{aligned}\right\} \tag{6.6}$$

where

$$L=\sqrt[4]{\frac{D}{k}}$$

l = plate length, b = plate width:

$$\alpha=\sqrt{\frac{k}{2t}}.$$

Introducting the notations.

$$r_1^2=n^2\pi^2\frac{l^2}{b^2},\quad r_0^2=\frac{l^2}{2\alpha^2L^4}. \tag{6.7}$$

we obtain:

$$r^2=r_1^2+r_0^2,$$

It is seen that (6.4) has the same form as the differential equation of the bending of a beam (1.8) of Chapter II, differing only in the values of the constants r^2 and s^4. It follows that methods similar to those used for two-dimensional analysis of beams can be applied to this problem (cf. sections 2, 3, 6, of Chapter II).

§ 7. SOLVING THE DIFFERENTIAL EQUATION OF THE BENDING OF A PLATE BY THE METHOD OF INITIAL PARAMETERS

When an arbitrary external load is applied to the plate, equation (6.4) is most simply solved by the method of initial parameters.

The general solution is then:

$$\left.\begin{aligned} W(\eta) &= K_{WW}W_0+K_{W\varphi}\varphi_0+K_{WM}M_0+K_{WN}N_0-F_W,\\ \varphi(\eta) &= K_{\varphi W}W_0+K_{\varphi\varphi}\varphi_0+K_{\varphi M}M_0+K_{\varphi N}N_0-F_\varphi,\\ M(\eta) &= K_{MW}W_0+K_{M\varphi}\varphi_0+K_{MM}M_0+K_{MN}N_0-F_M,\\ N(\eta) &= K_{NW}W_0+K_{N\varphi}\varphi_0+K_{NM}M_0+K_{NN}N_0-F_N, \end{aligned}\right\} \tag{7.1}$$

where K_{WW}, $K_{W\varphi}, \ldots, K_{NN}$ = influence functions; W_0, φ_0, M_0, N_0 = generalized deflection, generalized slope, generalized bending moment, generalized shearing force respectively (at y = 0); $F_W, \ldots, F_N$ = load functions.

In order to determine the influence functions in (7.1) we have to find the solution of the homogeneous equation corresponding to (6.4), which is:

$$W(\eta)=C_1\Phi_1+C_2\Phi_2+C_3\Phi_3+C_4\Phi_4, \tag{7.2}$$

where $C_1, C_2, \ldots, C_4$ = integration constants; $\Phi_1, \ldots, \Phi_4$ = functions depending on the roots of the auxiliary equation, i.e., on the coefficients r^2 and s^4.

The case:

$$s > r.$$

is that most frequently met.

The functions $\Phi_1, \ldots, \Phi_4$ are in this case (cf. Table 3, p. 51):

$$\left.\begin{aligned} \Phi_1 &= \operatorname{sh} \bar{\alpha}\eta \cos \bar{\beta}\eta, & \Phi_2 &= \operatorname{ch} \bar{\alpha}\eta \cos \bar{\beta}\eta, \\ \Phi_3 &= \operatorname{ch} \bar{\alpha}\eta \sin \bar{\beta}\eta, & \Phi_4 &= \operatorname{sh} \bar{\alpha}\eta \sin \bar{\beta}\eta, \end{aligned}\right\} \tag{7.3}$$

$$\bar{\alpha} = \sqrt{\frac{s^2 + r^2}{2}}, \quad \beta = \sqrt{\frac{s^2 - r^2}{2}}. \tag{7.3'}$$

Since W is a function of the dimensionless coordinate $\eta = \frac{y}{l}$, the generalized slope φ is:

$$\varphi = \frac{dW}{dy} = \frac{1}{l} W'. \tag{7.4}$$

We then obtain from (3.5), (3.6), and (6.1) the generalized bending moment M and the generalized shearing force N:

$$\left.\begin{aligned} M &= -\frac{a}{l^2}\left(W'' - \mu \frac{bl^2}{a} W\right), \\ N &= -\frac{a}{l^3}\left\{W''' - \left[(2-\mu)\frac{bl^2}{a} + \frac{2\rho^0 l^2}{a}\right] W'\right\}, \end{aligned}\right\} \tag{7.5}$$

where μ = Poisson's ratio for material of plate; a, b, and ρ^0 are given by (6.2).

Substituting (6.2) and (6.7) in (7.5), we obtain:

$$\left.\begin{aligned} M &= -\frac{a}{l^2}(W'' - \mu r_1^2 W), \\ N &= -\frac{a}{l^3}\{W''' - [(2-\mu) r_1^2 + 2r_0^2] W'\}. \end{aligned}\right\} \tag{7.6}$$

We then obtain from (7.2), (7.3), (7.4), and (7.6):

$$\left.\begin{aligned} W &= C_1\Phi_1 + C_2\Phi_2 + C_3\Phi_3 + C_4\Phi_4, \\ l\varphi &= C_1(\bar{\alpha}\Phi_2 - \bar{\beta}\Phi_4) + C_2(\bar{\alpha}\Phi_1 - \bar{\beta}\Phi_3) + C_3(\bar{\alpha}\Phi_4 + \bar{\beta}\Phi_2) + \\ &\quad + C_4(\alpha\Phi_3 + \bar{\beta}\Phi_1) \\ \frac{l^2}{a}M &= -C_1\{[(1-\mu) r_1^2 + r_0^2]\Phi_1 - 2\bar{\alpha}\bar{\beta}\Phi_3\} - C_2\{[(1-\mu) r_1^2 + \\ &\quad + r_0^2]\Phi_2 - 2\overline{\alpha\beta}\Phi_4\} - C_3\{[(1-\mu) r_1^2 + r_0^2]\Phi_5 + \\ &\quad + 2\overline{\alpha\beta}\Phi_1\} - C_4\{[(1-\mu) r_1^2 + r_0^2]\Phi_4 + 2\bar{\alpha}\bar{\beta}\Phi_2\} \\ \frac{l^3}{a}N &= C_1\{\bar{\alpha}[s^2 - \mu r_1^2]\Phi_2 + \bar{\beta}[s^2 + \mu r_1^2]\Phi_4\} + \\ &\quad + C_2\{\bar{\alpha}[s^2 - \mu r_1^2]\Phi_1 + \bar{\beta}[s^2 + \mu r_1^2]\Phi_3\} - C_3\{\bar{\beta}[s^2 + \mu r_1^2]\Phi_2 - \\ &\quad - \bar{\alpha}[s^2 - \mu r_1^2]\Phi_4\} - C_4\{\bar{\beta}[s^2 + \mu r_1^2]\Phi_1 - \bar{\alpha}[s^2 - \mu r_1^2]\Phi_3\}. \end{aligned}\right\} \tag{7.7}$$

For $\eta = 0$ these expressions become:

$$\left.\begin{aligned} W_0 &= C_2, \\ l\varphi_0 &= \alpha C_1 + \bar{\beta} C_3, \\ \frac{l^2}{a} M_0 &= -C_2[(1-\mu) r_1^2 + r_0^2] - 2\alpha\beta C_4, \\ \frac{l^3}{a} N_0 &= C_1 \bar{\alpha}[s^2 - \mu r_1^2] - C_3 \bar{\beta}[s^2 + \mu r_1^2]. \end{aligned}\right\} \quad (7.8)$$

Solving (7.8) for C_1, C_2, C_3, C_4 yields:

$$\left.\begin{aligned} C_1 &= \frac{1}{2\bar{\alpha}\bar{\beta}s^2}\left[(s^2 + \mu r_1^2)\bar{\beta} l\varphi_0 + \bar{\beta}\frac{l^3}{a} N_0\right], \\ C_2 &= W_0, \\ C_3 &= \frac{1}{2\bar{\alpha}\bar{\beta}s^2}\left[(s^2 - \mu r_1^2)\bar{\alpha} l\varphi_0 - \bar{\alpha}\frac{l^3}{a} N_0\right], \\ C_4 &= -\frac{1}{2\bar{\alpha}\bar{\beta}}\left\{[(1-\mu) r_1^2 + r_0^2] W_0 + \frac{l^2}{a} M_0\right\}, \end{aligned}\right\} \quad (7.9)$$

where $\bar{\alpha}$ and $\bar{\beta}$ are given by (7.3'), and s, r_1, and r_0 by (6.6) and (6.7).

TABLE 10.

	W_0	φ_0	M_0	N_0
W_η	$K_{WW} = \frac{1}{2\bar{\alpha}\bar{\beta}}\{2\bar{\alpha}\bar{\beta}\Phi_2 - [(1-\mu) r_1^2 + r_0^2]\Phi_4\}$	$K_{W\varphi} = \frac{l}{2\bar{\alpha}\bar{\beta}s^2}[(s^2 + \mu r_1^2)\bar{\beta}\Phi_1 + (s^2 - \mu r_1^2)\bar{\alpha}\Phi_3]$	$K_{WM} = -\frac{l^2}{2a\bar{\alpha}\bar{\beta}}\Phi_4$	$K_{WN} = \frac{l^3}{2a\bar{\alpha}\bar{\beta}s^2}\times(\bar{\beta}\Phi_1 - \bar{\alpha}\Phi_3)$
φ_η	$K_{\varphi W} = \frac{1}{2l\bar{\alpha}\bar{\beta}}[(s^2 + \mu r_1^2)\bar{\beta}\Phi_1 - (s^2 - \mu r_1^2)\bar{\alpha}\Phi_3]$	$K_{\varphi\varphi} = \frac{1}{2\bar{\alpha}\bar{\beta}}\{2\bar{\alpha}\bar{\beta}\Phi_2 + [(1-\mu) r_1^2 + r_0^2]\Phi_4\}$	$K_{\varphi M} = -\frac{l}{2a\bar{\alpha}\bar{\beta}}\times(\bar{\alpha}\Phi_3 + \bar{\beta}\Phi)$	$K_{\varphi N} = K_{WM}$
M_η	$K_{MW} = \frac{a}{2l^2\bar{\alpha}\bar{\beta}}\{s^4 - \mu r_1^2[(2-\mu) r_1^2 + 2r_0^2]\}\Phi_4$	$K_{M\varphi} = -\frac{a}{2\bar{\alpha}\bar{\beta}s^2 l}\{\bar{\beta}\{(s^2 + \mu r_1^2)\times[(1+\mu) r_1^2 + r_0^2] + (s^2 - \mu r_1^2)[s^2 + r^2]\}\Phi_1 + \alpha\{(s^2 - \mu r_1^2)[(1-\mu) r_1^2 + r_0^2] - (s^2 + \mu r_1^2)[s^2 - r^2]\}\Phi_3\}$	$K_{MM} = K_{\varphi\varphi}$	$K_{MN} = K_{W\varphi}$
N_η	$K_{NW} = \frac{a}{2\bar{\alpha}\bar{\beta}l^3}\{\bar{\beta}[(s^2 + r^2)(s^2 - \mu r_1^2) + [(1-\mu) r_1^2 + r_0^2](s^2 + \mu r_1^2)]\Phi_1 + \bar{\alpha}[(s^2 - r^2)(s^2 + \mu r_1^2) - [(1-\mu) r_1^2 + r_0^2](s^2 - \mu r_1^2)]\Phi_3\}$	$K_{N\varphi} = K_{MW}$	$K_{NM} = K_{\varphi W}$	$K_{NN} = K_{WW}$

By substituting (7.9) in (7.7) the solution of the homogeneous equation corresponding to (6.4) can be expressed through the initial parameters and influence functions (the functions $F_W, \ldots, F_N$ do not appear in the expressions obtained). The influence functions are given in Table 10.

In order to take into account the external load, represented by the right-side of (6.4), we subtract from the expressions (7.7) terms corresponding to the functions $F_W, \ldots, F_N$, which depend on the applied load and its distribution over the plate. This dependence was discussed in section 3 of Chapter II. One example of the application of this method will be given here.

Let a load as shown in Figure 82 act on the plate. It is seen that in part $0<\eta<t_1$ the homogeneous differential equation is applicable so that all unknowns are determined by the initial parameters

$$\left.\begin{aligned}W(\eta) &= K_{WW}(\eta)W_0+K_{W\varphi}(\eta)\varphi_0+K_{WM}(\eta)M_0+K_{WN}(\eta)N_0,\\ \varphi(\eta) &= K_{\varphi W}(\eta)W_0+K_{\varphi\varphi}(\eta)\varphi_0+K_{\varphi M}(\eta)M_0+K_{\varphi N}(\eta)N_0,\\ M(\eta) &= K_{MW}(\eta)W_0+K_{M\varphi}(\eta)\varphi_0+K_{MM}(\eta)M_0+K_{MN}(\eta)N_0,\\ N(\eta) &= K_{NM}(\eta)W_0+K_{N\varphi}(\eta)\varphi_0+K_{NM}(\eta)M_0+K_{NN}(\eta)N_0.\end{aligned}\right\} \qquad (7.10)$$

For $t_1<\eta<t_2$ the following expressions have to be subtracted from the respective equations (7.10):

$$\left.\begin{aligned}F_W &= G_1K_{WN}(\eta-t_1), & F_M &= G_1K_{MN}(\eta-t_1),\\ F_\varphi &= G_1K_{\varphi N}(\eta-t_1), & F_N &= G_1K_{NN}(\eta-t_1);\end{aligned}\right\} \qquad (7.11)$$

where

$$G_1=\sum P_c\chi(c)=P_1\chi(c_1)+P_2\chi(c_2),$$

and $\chi(c)$ = value of $\chi(x)$ at the point of application of the concentrated force.

For $t_2<\eta<t_3$ the load functions are:

$$\left.\begin{aligned}F_W &= G_1K_{WN}(\eta-t_1)+G_3K_{WN}(\eta-t_2),\\ F_\varphi &= G_1K_{\varphi N}(\eta-t_1)+G_3K_{\varphi N}(\eta-t_2),\\ F_M &= G_1K_{MN}(\eta-t_1)+G_3K_{MN}(\eta-t_2),\\ F_N &= G_1K_{NN}(\eta-t_1)+G_3K_{NN}(\eta-t_2),\end{aligned}\right\} \qquad (7.12)$$

where

$$G_3=P_3\chi(c_3).$$

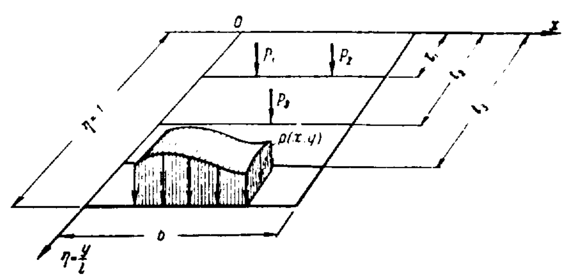


FIGURE 82.

For $t_3 < \eta < 1$ the load functions are:

$$\left.\begin{aligned}
F_W &= G_1K_{WN}(\eta - t_1) + G_3K_{WN}(\eta - t_2) + \\
&\quad + \int_0^\eta G_4(t)\,K_{WN}(\eta - t)\,dt, \\
F_\varphi &= G_1K_{\varphi N}(\eta - t_1) + G_3K_{\varphi N}(\eta - t_2) + \\
&\quad + \int_0^\eta G_4(t)\,K_{\varphi N}(\eta - t)\,dt, \\
F_M &= G_1K_{MN}(\eta - t_1) + G_3K_{MN}(\eta - t_2) + \\
&\quad + \int_0^\eta G_4(t)\,K_{MN}(\eta - t)\,dt, \\
F_N &= G_1K_{NN}(\eta - t_1) + G_3K_{NN}(\eta - t_2) + \\
&\quad + \int_0^\eta G_4(t)\,K_{NN}(\eta - t)\,dt,
\end{aligned}\right\} \tag{7.13}$$

where

$$G_4 = \int_0^b p(x, \eta)\,\chi(x)\,dx.$$

§ 8. DETERMINATION OF INITIAL PARAMETERS. CALCULATION OF BENDING MOMENTS AND SHEARING FORCES

Since the origin can be in any plate cross section, two of the four initial parameters W_0, φ_0, M_0, N_0 in (7.1) are usually determined directly from the boundary conditions (cf. sections 3, 6 of Chapter II). The two other parameters are determined, irrespective of the applied external load, by solving simultaneously two equations written for a different cross section ($\eta = \text{const}$) of the plate. This will be illustrated by several examples.

Simply supported lateral plate edges

The load applied along the η axis will be divided into symmetrical and antisymmetrical components. The origin of coordinates lies in the center section of the plate (Figure 83).

The boundary conditions for symmetrical loading (Figure 83, a), are:

$$\eta = \frac{y}{l} = 0 \qquad \varphi_0 = 0, \quad N_0 = 0. \tag{8.1}$$

the general solution then becomes:

$$\left.\begin{aligned}
W(\eta) &= W_0K_{WW}(\eta) + M_0K_{WM}(\eta) - F_W(\eta), \\
\varphi(\eta) &= W_0K_{\varphi W}(\eta) + M_0K_{\varphi M}(\eta) - F_\varphi(\eta), \\
M(\eta) &= W_0K_{MW}(\eta) + M_0K_{MM}(\eta) - F_M(\eta), \\
N(\eta) &= W_0K_{NW}(\eta) + M_0K_{NM}(\eta) - F_N(\eta).
\end{aligned}\right\} \tag{8.2}$$

where the functions $F_{W,\eta} \ldots, F_N$ correspond to the load on only one half of the plate ($\eta > 0$). The parameters W_0 and M_0 are determined from the boundary conditions at the lateral edge:

$$\text{at} \quad \eta = \frac{1}{2}\left(\eta = -\frac{1}{2}\right): \; W = 0, \; M = 0. \tag{8.3}$$

Substitution of (8.3) in (8.2) yields:

$$\left.\begin{aligned} W\left(\tfrac{1}{2}\right) &= W_0 K_{WW}\left(\tfrac{1}{2}\right) + M_0 K_{WM}\left(\tfrac{1}{2}\right) - F_W\left(\tfrac{1}{2}\right) = 0, \\ M\left(\tfrac{1}{2}\right) &= W_0 K_{MW}\left(\tfrac{1}{2}\right) + M_0 K_{MM}\left(\tfrac{1}{2}\right) - F_M\left(\tfrac{1}{2}\right) = 0. \end{aligned}\right\} \tag{8.4}$$

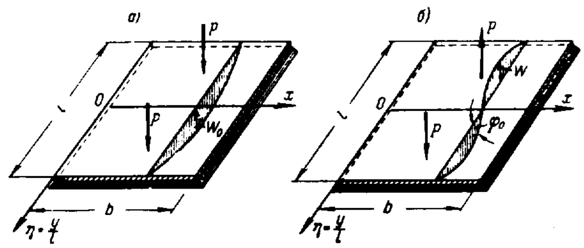


FIGURE 83.

The solution of system (8.4) is:

$$\left.\begin{aligned} W_0 &= \frac{F_W\left(\frac{1}{2}\right) K_{MM}\left(\frac{1}{2}\right) - F_M\left(\frac{1}{2}\right) K_{WM}\left(\frac{1}{2}\right)}{K_{WW}\left(\frac{1}{2}\right) K_{MM}\left(\frac{1}{2}\right) - K_{WM}\left(\frac{1}{2}\right) K_{MW}\left(\frac{1}{2}\right)}, \\ M_0 &= \frac{F_M\left(\frac{1}{2}\right) K_{WW}\left(\frac{1}{2}\right) - F_W\left(\frac{1}{2}\right) K_{MW}\left(\frac{1}{2}\right)}{K_{WW}\left(\frac{1}{2}\right) K_{MM}\left(\frac{1}{2}\right) - K_{WM}\left(\frac{1}{2}\right) K_{MW}\left(\frac{1}{2}\right)}, \end{aligned}\right\} \tag{8.5}$$

For antisymmetrical loading (Figure 83, b), the boundary conditions at $\eta = 0$ are:

$$\text{at} \quad \eta = \frac{y}{l} = 0 \; : \; W_0 = 0, \quad M_0 = 0. \tag{8.6}$$

The general solution for this case is:

$$\left.\begin{aligned} W(\eta) &= \varphi_0 K_{W\varphi}(\eta) + N_0 K_{WN}(\eta) - F_W(\eta), \\ \varphi(\eta) &= \varphi_0 K_{\varphi\varphi}(\eta) + N_0 K_{\varphi N}(\eta) - F_\varphi(\eta), \\ M(\eta) &= \varphi_0 K_{M\varphi}(\eta) + N_0 K_{MN}(\eta) - F_M(\eta), \\ N(\eta) &= \varphi_0 K_{N\varphi}(\eta) + N_0 K_{NN}(\eta) - F_N(\eta), \end{aligned}\right\} \tag{8.7}$$

where the functions $F_W(\eta), \ldots, F_N(\eta)$ again correspond to the load on only one half of the plate.

The boundary conditions (8.3) yield:

$$\left.\begin{aligned} W\left(\frac{1}{2}\right) &= \varphi_0 K_{W\varphi}\left(\frac{1}{2}\right) + N_0 K_{WN}\left(\frac{1}{2}\right) - F_W\left(\frac{1}{2}\right) = 0, \\ M\left(\frac{1}{2}\right) &= \varphi_0 K_{M\varphi}\left(\frac{1}{2}\right) + N_0 K_{MN}\left(\frac{1}{2}\right) - F_M\left(\frac{1}{2}\right) = 0, \end{aligned}\right\} \quad (8.8)$$

whence

$$\left.\begin{aligned} \varphi_0 &= \frac{F_W\left(\frac{1}{2}\right) K_{MN}\left(\frac{1}{2}\right) - F_M\left(\frac{1}{2}\right) K_{WN}\left(\frac{1}{2}\right)}{K_{W\varphi}\left(\frac{1}{2}\right) K_{MN}\left(\frac{1}{2}\right) - K_{WN}\left(\frac{1}{2}\right) K_{M\varphi}\left(\frac{1}{2}\right)}, \\ N_0 &= \frac{F_M\left(\frac{1}{2}\right) K_{W\varphi}\left(\frac{1}{2}\right) - F_W\left(\frac{1}{2}\right) K_{M\varphi}\left(\frac{1}{2}\right)}{K_{W\varphi}\left(\frac{1}{2}\right) K_{MN}\left(\frac{1}{2}\right) - K_{WN}\left(\frac{1}{2}\right) K_{M\varphi}\left(\frac{1}{2}\right)}. \end{aligned}\right\} \quad (8.9)$$

Built-in lateral plate edges

In this case (Figure 84), the boundary conditions at $\eta = 0$ for symmetrical and antisymmetrical loading are given by (8.1) and (8.6) respectively. The solution of the problem is given by (8.2) for symmetrical and (8.7) for antisymmetrical loading.

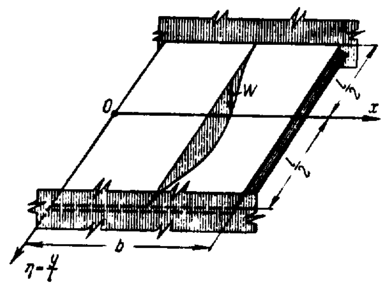


FIGURE 84.

The boundary conditions at the built-in lateral plate edges are:

$$\eta = \frac{1}{2} \ \left(\eta = -\frac{1}{2}\right) \quad W = 0, \quad \varphi = 0. \qquad (8.10)$$

The initial parameters are determined from (8.10) and (8.2) or (8.7): for symmetrical loading:

$$\left.\begin{aligned} W_0 &= \frac{F_W K_{\varphi M} - F_\varphi K_{WM}}{K_{WW} K_{\varphi M} - K_{WM} K_{\varphi W}}, \\ M_0 &= \frac{F_\varphi K_{WW} - F_W K_{\varphi W}}{K_{WW} K_{\varphi M} - K_{WM} K_{\varphi W}}; \end{aligned}\right\} \quad (8.11)$$

for antisymmetrical loading

$$\left.\begin{aligned}\varphi_0&=\frac{F_W K_{\varphi N}-F_\varphi K_{WN}}{K_{W\varphi}K_{\varphi N}-K_{WN}K_{\varphi\varphi}},\\ N_0&=\frac{F_\varphi K_{W\varphi}-F_W K_{\varphi\varphi}}{K_{W\varphi}K_{\varphi N}-K_{WN}K_{\varphi\varphi}},\end{aligned}\right\}\tag{8.12}$$

where

$$K_{\varphi M},\ K_{WM},\ \ldots,\ K_{\varphi\varphi},\ K_{\varphi N},\ F_W,\ F_\varphi$$

are the values of the corresponding functions at $\eta=\frac{1}{2}$.

Free lateral plate edges

The boundary conditions at the free plate edges are (Figure 85):

$$\left.\begin{aligned}W\left(\tfrac{1}{2}\right)&=W_f,\\ N\left(\tfrac{1}{2}\right)&=S_f,\\ M\left(\tfrac{1}{2}\right)&=0,\end{aligned}\right\}\tag{8.13}$$

where W_f = vertical displacement of foundation at $\eta=\frac{y}{l}=\frac{1}{2}$, S_f = generalized shearing force exerted by foundation in this section.

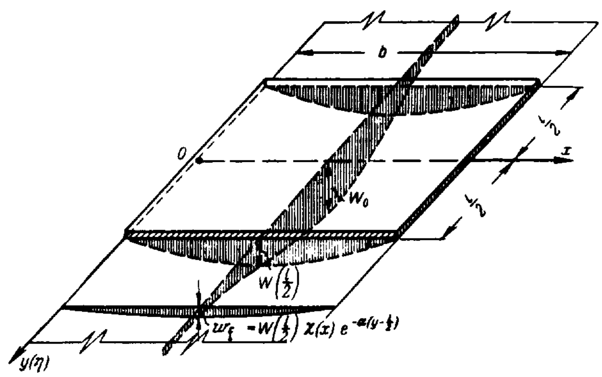


FIGURE 85.

The first condition (8.13) is purely geometrical and expresses the equality of the vertical depressions of plate and foundation surface. The second and third conditions are statical equilibrium conditions, similar to the corresponding conditions for the free end of a beam of finite length on an elastic single-layer foundation (cf. section 6 of Chapter II).

The vertical displacements of the free foundation surface ($|\eta| > \frac{1}{2}$) can be approximated by an exponential function (Figure 85):

$$w_{\mathrm{f}} = W_{\mathrm{f}}(y)\chi(x) = W_{\mathrm{f}}\left(\frac{1}{2}\right)\chi(x)\,e^{-\alpha\left(y-\frac{l}{2}\right)}, \tag{8.14}$$

where $\alpha = \sqrt{\frac{k}{2t}}$, and $\chi(x)$ is given by (6.1). The first boundary condition (8.13) is then satisfied by (8.14).

The generalized shearing force S_{f} in the free foundation, determined by the work done by the shearing stresses τ_{zy} over the corresponding virtual displacements, is given by (3.4). Since in this case $\chi(x)$ differs from zero only when $0 < x < b$, (3.4) reduces to:

$$S_{\mathrm{f}} = 2t\sum_{k=1}^{n} W'_{\mathrm{f}}\int_0^b \chi_k\chi_i\,dx. \tag{8.15}$$

Substituting (6.1) in (8.15) and taking (8.14) into account, we obtain for $y > \frac{l}{2}$

$$S_{\mathrm{f}} = -\alpha tbW_{\mathrm{f}}\left(\frac{1}{2}\right)e^{-\alpha\left(y-\frac{l}{2}\right)}. \tag{8.16}$$

The values of this expression at $y = \frac{l}{2}$ $\left(\eta = \frac{1}{2}\right)$ is:

$$S_{\mathrm{f}} = -\alpha tbW_{\mathrm{f}}\left(\frac{1}{2}\right).$$

Since

$$W\left(\frac{1}{2}\right) = W_{\mathrm{f}}\left(\frac{1}{2}\right),$$

we can write (8.13) as follows:

$$\left.\begin{aligned} N\left(\tfrac{1}{2}\right) &= -\alpha tbW\left(\tfrac{1}{2}\right),\\ M\left(\tfrac{1}{2}\right) &= 0. \end{aligned}\right\} \tag{8.17}$$

For symmetrical loading, the general solution is as before (cf. (8.2):

$$\left.\begin{aligned} W(\eta) &= W_0K_{WW}(\eta) + M_0K_{WM}(\eta) - F_W(\eta),\\ \varphi(\eta) &= W_0K_{\varphi W}(\eta) + M_0K_{\varphi M}(\eta) - F_\varphi(\eta),\\ M(\eta) &= W_0K_{MW}(\eta) + M_0K_{MM}(\eta) - F_M(\eta),\\ N(\eta) &= W_0K_{NW}(\eta) + M_0K_{NM}(\eta) - F_N(\eta). \end{aligned}\right\} \tag{8.18}$$

Substitution of (8.17) in (8.18) yields:

$$\left.\begin{aligned} W_0 &= \frac{(K_{NM} + \alpha tbK_{WM})F_M - K_{MM}(F_N + \alpha tbF_W)}{K_{MW}(K_{NM} + \alpha tbK_{WM}) - K_{MM}(K_{NW} + \alpha tbK_{WW})},\\ M_0 &= \frac{K_{MW}(F_N + \alpha tbF_W) - (K_{NW} + \alpha tbK_{WW})F_M}{K_{MW}(K_{NM} + \alpha tbK_{WM}) - K_{MM}(K_{NW} + \alpha tbK_{WW})}. \end{aligned}\right\} \tag{8.19}$$

For antisymmetrical loading we obtain:

$$\left.\begin{aligned} W(\eta) &= \varphi_0 K_{W\varphi}(\eta) + N_0 K_{WN}(\eta) - F_W(\eta), \\ \varphi(\eta) &= \varphi_0 K_{\varphi\varphi}(\eta) + N_0 K_{\varphi N}(\eta) - F_\varphi(\eta), \\ M(\eta) &= \varphi_0 K_{M\varphi}(\eta) + N_0 K_{MN}(\eta) - F_M(\eta), \\ N(\eta) &= \varphi_0 K_{N\varphi}(\eta) + N_0 K_{NN}(\eta) - F_N(\eta). \end{aligned}\right\} \tag{8.20}$$

Substitution of (8.17) in (8.20) yields:

$$\left.\begin{aligned} \varphi_0 &= \frac{(K_{NN} + \alpha t b K_{WN}) F_M - K_{MN}(F_N + \alpha t b F_W)}{K_{M\varphi}(K_{NN} + \alpha t b K_{WN}) - K_{MN}(K_{N\varphi} + \alpha t b K_{W\varphi})}, \\ N_0 &= \frac{K_{M\varphi}(F_N + \alpha t b F_W) - (K_{N\varphi} + \alpha t b K_{W\varphi}) F_M}{K_{M\varphi}(K_{NN} + \alpha t b K_{WN}) - K_{MN}(K_{N\varphi} + \alpha t b K_{W\varphi})}. \end{aligned}\right\} \tag{8.21}$$

The values of $K_{NN}(\eta)\ldots, K_{W\varphi}(\eta)$ $F_W(\eta)$ $F_N(\eta)$ in (8.19) and (8.21) are taken for $\eta = \frac{1}{2}$ $\left(y = \frac{l}{2}\right)$. As before:

$$\alpha = \sqrt{\frac{k}{2t}}, \quad k = \frac{E_0}{(1 - \nu_0^2)} \int_0^H \psi'^2 \, dz, \quad t = \frac{F_0}{4(1 + \nu_0)} \int_0^H \psi^2 \, dz, \tag{8.22}$$

where

$$E_0 = \frac{E_s}{1 - \nu_s^2}, \quad \nu_0 = \frac{\nu_s}{1 - \nu_s}.$$

Different boundary conditions at the lateral edges

If the boundary conditions at the two lateral edges differ, the x axis is placed along one of these edges (Figure 86). The initial parameters can then be determined in exactly the same way as for a symmetrical plate. From (8.17), we obtain for the free edge $\eta = 0$:

$$M_0 = 0, \quad N_0 = \alpha t b W_0. \tag{8.23}$$

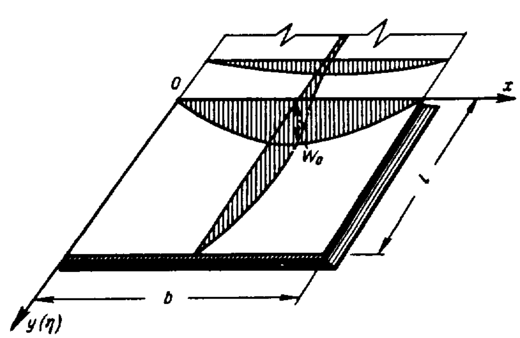


FIGURE 86.

The general solution is then:

$$\left.\begin{aligned} W(\eta) &= (K_{WW} + \alpha tbK_{WN})\,W_0 + K_{W\varphi}\varphi_0 - F_W,\\ \varphi(\eta) &= (K_{\varphi W} + \alpha tbK_{\varphi N})\,W_0 + K_{\varphi\varphi}\varphi_0 - F_\varphi,\\ M(\eta) &= (K_{MW} + \alpha tbK_{MN})\,W_0 + K_{M\varphi}\varphi_0 - F_M,\\ N(\eta) &= (K_{NM} + \alpha tbK_{NN})\,W_0 + K_{N\varphi}\varphi_0 - F_N. \end{aligned}\right\} \tag{8.24}$$

The values of W_0 and φ_0 can now be determined from the boundary conditions at $\eta = 1$ $(y = l)$. Thus, if this end is simply supported the boundary conditions become:

$$\text{at } \eta = 1: \qquad W = 0, \quad M = 0. \tag{8.25}$$

From (8.24) and (8.25) we then obtain:

$$\left.\begin{aligned} W_0 &= \frac{K_{M\varphi}F_W - K_{W\varphi}F_M}{K_{M\varphi}(K_{WW} + \alpha tbK_{WN}) - K_{W\varphi}(K_{MW} + \alpha tbK_{MN})},\\ \varphi_0 &= \frac{(K_{WW} + \alpha tbK_{WN})\,F_M - (K_{MW} + \alpha tbK_{MN})\,F_W}{K_{M\varphi}(K_{WW} + \alpha tbK_{WN}) - K_{M\varphi}(K_{MW} + \alpha tbK_{MN})}, \end{aligned}\right\} \tag{8.26}$$

where the functions $K_{M\varphi}, K_{W\varphi}, \ldots, F_W, F_M$ are taken at $\eta = 1$ $(y = l)$. In this case the functions F_W, F_M correspond to the load on the entire plate $(0 \leqslant \eta \leqslant 1)$.

Calculating the bending moments and shearing forces

When the generalized plate deflection $W(\eta)$ has been determined, the bending moments and shearing forces can be found from (2.28). Noting that $W(\eta)$ is a function of $\eta = \frac{y}{l}$, and taking (6.1) into account, we obtain for each term of (2.1):

$$\left.\begin{aligned} M_x &= -D\left[\frac{\mu}{l^2} W''(\eta) - \left(\frac{n\pi}{b}\right)^2 W(\eta)\right]\sin\frac{n\pi x}{b},\\ M_y &= -D\left[\frac{1}{l^2} W''(\eta) - \mu\left(\frac{n\pi}{b}\right)^2 W(\eta)\right]\sin\frac{n\pi x}{b},\\ H = H_x &= -H_y = -D\,\frac{1-\mu}{l}\,\frac{n\pi}{b}\,W'(\eta)\cos\frac{n\pi x}{b},\\ N_x &= -D\left[\frac{1}{l^2}\,\frac{n\pi}{b}\,W''(\eta) - \left(\frac{n\pi}{b}\right)^3 W(\eta)\right]\cos\frac{n\pi x}{b},\\ N_y &= -D\left[\frac{1}{l^3} W'''(\eta) - \frac{1}{l}\left(\frac{n\pi}{b}\right)^2 W'(\eta)\right]\sin\frac{n\pi x}{b}. \end{aligned}\right\} \tag{8.27}$$

§ 9. CYLINDRICAL BENDING AND TORSION OF A NARROW PLATE. THREE-DIMENSIONAL BEAM ANALYSIS

1

Consider a narrow rectangular plate loaded symmetrically with respect to the y axis (Figure 87).

If we assume that the cross section of this plate (which in the general case is of varying thickness) is not deformed, only the translational displacement

$$\chi_0 = 1. \tag{9.1}$$

remains from all the possible displacements χ_i of an elementary strip of width $dy = 1$. The coefficients (2.26) entering in (2.25) then become:

$$\left.\begin{aligned} a_{00} &= \sum D_m b_m, & b_{00} &= c_{00} = 0, \\ p^0_{00} &= t\left(b + \frac{1}{\alpha}\right), & s^0_{00} &= kb + 4\alpha t, \end{aligned}\right\} \tag{9.2}$$

where

$$D_m = \frac{Eh_m^3}{12(1-\mu^2)}$$

is the flexural rigidity of the plate for a part of length b_m of the cross section (Figure 88).

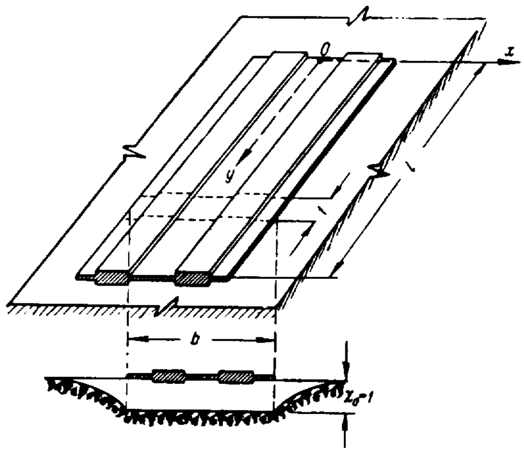


FIGURE 87.

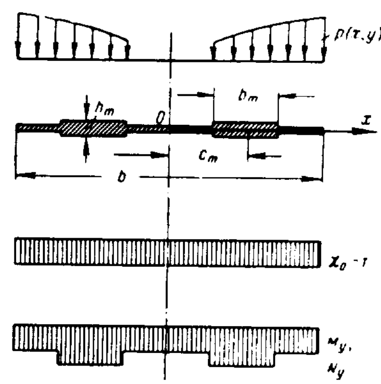


FIGURE 88.

Since

$$D_m b_m = \frac{EJ_m}{1-\mu^2},$$

we obtain

$$a_{00} = \frac{E}{1-\mu^2}\sum J_m = \frac{EJ}{1-\mu^2}, \tag{9.3}$$

where J is the total moment of inertia of the cross section relative to the x axis.

Substituting (9.2) and (9.3) in (2.25), we obtain:

$$\begin{aligned} W^{IV} - 2r^2W'' + s^4W = \\ = \frac{G(1-\mu^2)l}{EJ}, \end{aligned} \tag{9.4}$$

where

$$\left.\begin{aligned} r^2 &= \frac{1-\mu^2}{EJ} t\left(b+\frac{1}{\alpha}\right), \\ s^4 &= \frac{1-\mu^2}{EJ}(kb+4\alpha t). \end{aligned}\right\} \tag{9.5}$$

In this case, the value of G is that of the actual load, as can be seen from (2.4) and (9.1).

The differential equation (9.4) of the cylindrical bending of a plate has the same form as the equation of the bending of a beam in the two-dimensional problem ((1.8) of Chapter II); it differs from it in that Poisson's ratio μ enters in (9.4). The values of the coefficients s^4 and r^2 are also different. Through the terms

$$\frac{1-\mu^2}{EJ} 4\alpha t, \quad \frac{1-\mu^2}{EJ}\frac{t}{\alpha}$$

entering in these coefficients, allowance is made for the fictitious reactions Q^{Φ} distributed over the longitudinal plate edges, i.e., for the three-dimensional state of stress in the elastic foundation.

We then obtain from (3.5) and (3.6):

$$\left.\begin{aligned} M &= -\frac{EJ}{1-\mu^2} W'', \\ N &= -\frac{EJ}{1-\mu^2} W''' + 2tb\left(1+\frac{1}{\alpha b}\right) W'. \end{aligned}\right\} \tag{9.6}$$

We can now integrate (9.4) by the methods of sections 2 and 3 of Chapter II. When the generalized deflection of the plate has been determined, the actual bending moments M_y and shearing forces N_y are obtained from (2.28 b, e), which in this case reduce to:

$$M_y = -DW'', \quad N_y = -DW'''. \tag{9.7}$$

It is seen from (9.7) that the bending moments M_y and shearing forces N_y in each cross section are proportional to the flexural rigidities D_m (Figure 89).

2

Consider now the same plate acted upon by an antisymmetrical load (Figure 89). Putting

$$\chi_1 = x, \quad \chi'(x) = 1, \tag{9.8}$$

where $\chi_1(x)$ has the dimension of length, the generalized deflection becomes:

$$W(y) = \frac{w(x, y)}{x}$$

which is a dimensionless magnitude, being the angle of twist of the plate:

$$\theta = \theta(y).$$

By (2.4) and (9.8), the generalized load is:

$$G = \int p(x, y)\, x\, dx + \sum p_c x_c = m(y) \tag{9.9}$$

and represents the twisting moment $m(y)$.

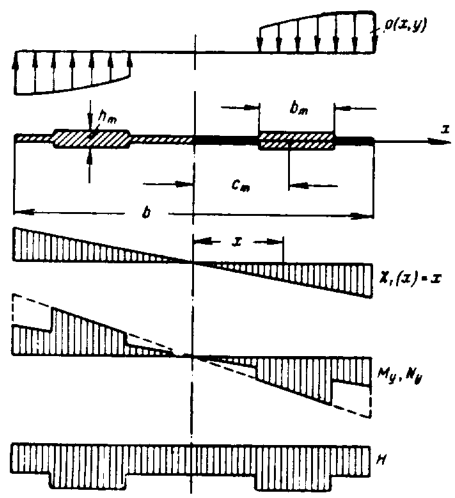


FIGURE 89.

The coefficients of (2.25) are again obtained from (2.26):

$$\left.\begin{aligned}
a_{11} &= \sum D_m \int x^2\, dx = \frac{EJ}{1-\mu^2}\rho^2,\\
b_{11} &= (1-\mu)\sum D_m b_m = \frac{EJ}{1+\mu},\\
c_{11} &= 0,\\
\rho_{11}^0 &= \frac{tb^3}{12}\left(1+\frac{3}{\alpha b}\right),\\
s_{11}^0 &= \frac{kb^3}{12}\left(1+\frac{12}{\alpha^2 b^2}+\frac{6}{\alpha b}\right),
\end{aligned}\right\} \tag{9.10}$$

where

$$\alpha = \sqrt{\frac{k}{2t}},$$

k and t = constants characterizing the compressive and shearing strains respectively of the elastic foundation, J = total moment of inertia of plate cross section relative to x axis; ρ = radius of inertia of rigidity diagram (Figure 89):

$$\rho^2 = \frac{1}{12}\,\frac{\sum D_m b_m (12c_m^2 + b_m^2)}{\sum D_m b_m} \tag{9.11}$$

[where c_m is the distance between the origin and the centroid of that part of the cross section whose rigidity is D_m].

For a plate of uniform thickness:

$$\rho^2 = \frac{b^2}{12}.$$

Equation (2.25) then becomes:

$$a_{11}\theta^{IV} - 2(b_{11} + \rho_{11}^0)\theta'' + s_{11}^0\theta - m = 0. \tag{9.12}$$

The generalized bending moment M and shearing force N are, by (3.5) and (3.6):

$$M = -\frac{EJ}{1-\mu^2}\rho^2\theta'', \tag{9.13}$$

$$N = -\frac{EJ}{1-\mu^2}\rho^2\theta''' + \frac{tb^3}{6}\left[1 + \frac{3}{ab}\right]\theta'. \tag{9.14}$$

where ρ^2 is defined by (9.11).

The actual bending moments M_y, torques H, and shearing forces N_y are by (2.28):

$$\left.\begin{aligned} M_y &= -Dx\theta'', \\ H &= -D(1-\mu)\theta', \\ N_y &= -Dx\theta'''. \end{aligned}\right\} \tag{9.15}$$

The distribution of M_y and N_y over the cross section $y = \text{const}$ is thus similar to that of $\chi_1 = x$ multiplied by the flexural rigidity D; while the twisting moments are directly proportional to the flexural rigidities D (Figure 89).

Together with (9.13) and (9.14), (9.12) determines the deformation of the plate, characterized by the presence of bending moments M_y in addition to the twisting moments H.

The generalized moment (9.13) represents in this case a bimoment, i.e., a system of normal stresses acting in the section $y = \text{const}$, statically equivalent to a zero force. The generalized shearing force determines the total twisting moment acting in the section $y = \text{const}$, due both to the shearing forces N_y and to the reactions of the elastic foundation; these are respectively given by the first and the second term of the right side of (9.14).

3

We can apply (9.4) to the three-dimensional problem of the bending of a beam by putting $\mu = 0$, and considering EJ as the rigidity of the beam. The free term G then represents the load per unit length.

Exactly as in the two-dimensional problem (cf. Chapter II), the beams can be classified as long, short, or rigid, depending on their rigidity.

Long beams acted upon by concentrated forces and moments can be analyzed by the method developed in section 4 of Chapter II. In the three-dimensional problems the generalized shearing force N entering in the boundary conditions is determined by the second equation (9.6), while r^2 and s^4 are given by (9.5).

Short beams acted upon by aribrary external loads are most simply analyzed by the method of initial parameters (section 3 of Chapter II, and section 7 of this chapter). When solving (9.4), the influence functions must be obtained from (9.5) and (9.6). The initial parameters are determined from the boundary conditions, given in generalized form, which, for free beam ends, correspond to (8.13).

In the case of rigid beams we can proceed directly from the equilibrium conditions of a beam acted upon by the known external load and by the reactions of the elastic foundation (cf. section 5 of Chapter II). Thus, for a symmetrical load, we obtain:

$$W(y) = C_0. \tag{9.16}$$

The reactions of the elastic foundation consist of the reactions q distributed over the surface supporting the beam, the reactions Q_l^Φ distributed along the longitudinal edges and the concentrated reactions T^Φ applied at the beam ends (Figure 90). The concentrated reaction T^Φ are introduced in order to make allowance for the effect of the deformation of the elastic foundation beyond the beam ends ($y < 0$, $y > l$) on the stresses in the beam.

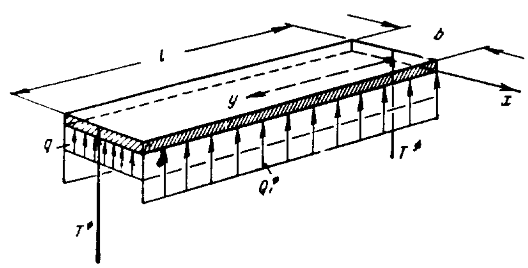


FIGURE 90.

From (2.17), (2.19), (9.1), and (9.16), we obtain:

$$q = kC_0, \quad Q_l^\Phi = 2\alpha t C_0. \tag{9.17}$$

The concentrated reactions T^Φ are obtained by assuming that for $y < 0$ and $y > l$, the vertical displacements of the surface of the elastic foundation decrease exponentially. Thus, for $y < 0$ we have (Figure 91):

$$\left.\begin{array}{l} \text{at } x \leqslant -\frac{b}{2} \\ \qquad w(x, y) = C_0 e^{\alpha\left(x+\frac{b}{2}\right)} e^{\alpha y}, \\ \text{at } -\frac{b}{2} \leqslant x \leqslant \frac{b}{2} \\ \qquad w(x, y) = C_0 e^{\alpha y}, \\ \text{at } x \geqslant \frac{b}{2} \\ \qquad w(x, y) = C_0 e^{-\alpha\left(x-\frac{b}{2}\right)} e^{\alpha y}. \end{array}\right\} \tag{9.18}$$

We assume that the work done by the reactions T^{Φ} over the displacement $\overline{C}_0 = 1$ is equal to the work done by all the internal forces in the elastic foundation in the region $y < 0$ over the virtual displacements (9.18) when $C_0 = 1$. We thus define the fictitious force T^{Φ} as the virtual work done by the normal and shearing stresses $\sigma_z, \tau_{zx}, \tau_{zy}$ in the elastic foundation in the region $y < 0$. We then obtain:

$$T^{\Phi} = C_0 (2\alpha tb + 3t). \tag{9.19}$$

Strictly speaking, the concentrated reactions T^{Φ} consist of the reactions Q^{Φ}_n distributed over the lateral edges of the beam and the concentrated reactions R^{Φ} at the corners (cf. section 10). However, since the beam is by definition rigid in the lateral direction, we can introduce the resultant concentrated force T^{Φ}.

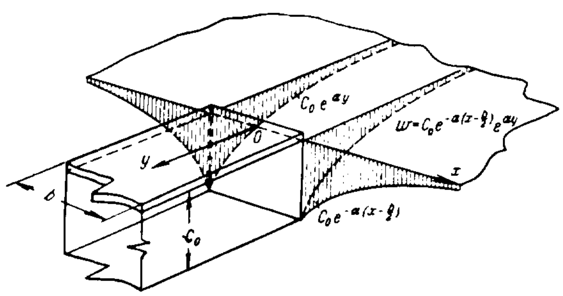


FIGURE 91.

The equilibrium condition of the beam is obtained by equating to zero the vertical projection of all forces acting on the beam. Taking (9.17) and (9.19) into account, we obtain:

$$[kbl + 4\alpha tl + 4\alpha tb + 6t]\, C_0 = P_0,$$

whence

$$C_0 = \frac{P_0}{[kbl + 4\alpha t(l+b) + 6t]}, \tag{9.20}$$

where P_0 = resultant vertical load acting on beam; l = beam length; b = beam width.

When C_0 has been obtained, the reactions of the foundation are found from (9.17) and (9.19); the bending moments and shearing forces are then determined by the known methods of the strength of materials.

The analysis of a rigid beam acted upon by an antisymmetrical load is performed similarly. If the origin of coordinates is placed at the beam center, the vertical displacements are:

$$W(y) = \theta_0 y,$$

where θ_0 is the slope of the beam, whose value can be determined by equating to zero the sum of all moments about the origin, acting on the beam:

$$\sum M_0 = 0$$

(cf. section 5 of Chapter II).

4

Consider a symmetrically loaded rigid beam. Assume that:

$$\psi(z) = \frac{\operatorname{sh} \gamma \frac{H-z}{b}}{\operatorname{sh} \gamma \frac{H}{b}}. \tag{9.21}$$

Substitution of (9.20) in (9.17) and (9.19) yields:

$$\left.\begin{aligned}
q &= \frac{P_0}{lb} \frac{1}{\left[1 + \frac{2}{\alpha b}\left(1 + \frac{b}{l}\right) + \frac{3}{\alpha^2 lb}\right]}, \\
Q_L^{\Phi} &= \frac{P_0}{b} \frac{1}{\alpha l \left[1 + \frac{2}{\alpha b}\left(1 + \frac{b}{l}\right) + \frac{3}{\alpha^2 lb}\right]}, \\
T^{\Phi} &= P_0 \frac{\left[1 + \frac{3}{2\alpha b}\right]}{\alpha l \left[1 + \frac{2}{\alpha b}\left(1 + \frac{b}{l}\right) + \frac{3}{\alpha^2 lb}\right]}.
\end{aligned}\right\} \tag{9.22}$$

When $\psi(z)$ is given by (9.21), the coefficient $\alpha = \sqrt{\frac{k}{2t}}$ entering in (9.22) (cf. (5.23), (5.24) of Chapter II) becomes:

$$\alpha = \frac{\gamma}{b} \sqrt{\frac{2}{1-\nu_0}} \sqrt{\frac{\operatorname{sh}\frac{\gamma H}{b} \cdot \operatorname{ch}\frac{\gamma H}{b} + \frac{\gamma H}{b}}{\operatorname{sh}\frac{\gamma H}{b} \cdot \operatorname{ch}\frac{\gamma H}{b} - \frac{\gamma H}{b}}}. \tag{9.23}$$

If the single-layer foundation is an elastic semi-infinite space $\left(\frac{H}{b} \to \infty\right)$, we obtain:

$$\left.\begin{aligned}
q &= \frac{P_0}{lb} \frac{1}{\left[1 + \frac{2}{\gamma} \sqrt{\frac{1-\nu_0}{2}}\left(1 + \frac{b}{l}\right) + \frac{3}{2\gamma^2} \frac{b}{l} \; 1-\nu_0)\right]}, \\
Q_L^{\Phi} &= \frac{P_0}{b} \frac{1}{\gamma \frac{l}{b} \sqrt{\frac{2}{1-\nu_0}} \left[1 + \frac{2}{\gamma} \sqrt{\frac{1-\nu_0}{2}}\left(1 + \frac{b}{l}\right) + \frac{3}{2\gamma^2} \frac{b}{l} (1-\nu_0)\right]}, \\
T^{\Phi} &= P_0 \frac{1 + \frac{3}{2\gamma} \sqrt{\frac{1-\nu_0}{2}}}{\gamma \frac{l}{b} \sqrt{\frac{2}{1-\nu_0}} \left[1 + \frac{2}{\gamma} \sqrt{\frac{1-\nu_0}{2}}\left(1 + \frac{b}{l}\right) + \frac{3}{2\gamma^2} \frac{b}{l} (1-\nu_0)\right]}.
\end{aligned}\right\} \tag{9.24}$$

Figures 92 and 93 show the dimensionless bending moments $\overline{M}$ for two cases of loading, obtained from (9.24) for $\gamma = 1.5$; $\nu_0 = 0.3$; $\frac{l}{b} = 5$, and $\frac{l}{b} = 10$. Results obtained by Gorbunov-Posadov for the two- and three-dimensional rigid-beam problem ($\frac{l}{b} = 10$), are also given:

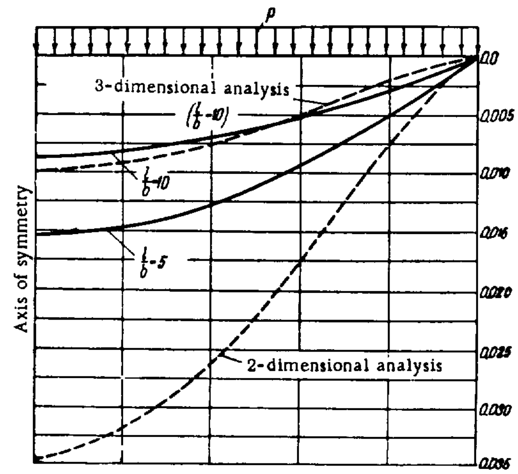


FIGURE 92.

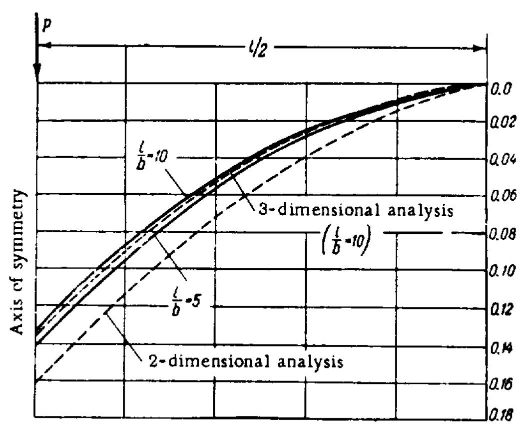


FIGURE 93.

The actual bending moments are:

$$M = \overline{M}pl^2,$$

for a uniformly distributed load, and

$$M = \overline{M}Pl.$$

for a concentrated load.

It is seen that the respective curves obtained by the two- and the three-dimensional analysis differ considerably: the maximum moment obtained by the two-dimensional analysis (for $\frac{l}{b} = 10$ and a uniformly distributed load) is almost 3.5 times the maximum moment obtained by the three-dimensional analysis.

It is also seen that the bending moments vary inversely with $\frac{l}{b}$: the wider the beam, the smaller the difference between the results obtained by two- and three-dimensional analysis.

A comparison of the results obtained by the method proposed and by Gorbunov-Posadov (for $\frac{l}{b} = 10$) shows that the difference between the maximum bending moments is relatively small (about 15% for a uniformly distributed load, and about 1.5% for a concentrated load).

§ 10. APPROXIMATE ANALYSIS OF A PLATE WITH FREE EDGES IN THE CASE OF SYMMETRICAL LOADING

1

Let a symmetrical load $p(x, y)$ be applied to a rectangular plate with free edges on a single-layer elastic foundation (Figure 94). The origin of coordinates is at the center of the plate. The differential equation of the bending of a plate on a single-layer elastic foundation is:

$$D\nabla^2\nabla^2 w - 2t\nabla^2 w + kw = p(x, y), \quad \text{[cf. (1.5)]} \qquad (10.1)$$

where $w(x, y)$ is the unknown deflection function of the plate, and

$$k = \frac{E_0}{(1 - \nu_0^2)} \int_0^H \psi'^2 dz, \quad t = \frac{E_0}{4(1 + \nu_0)} \int_0^H \psi^2 dz \quad \text{[cf. (1.6)]}. \qquad (10.2)$$

If $p(x, y)$ is distributed nearly uniformly over the plate a simple approximate solution can be obtained by writing:

$$w(x\ y) = C_0 + C_1 \cos\frac{\pi x}{2b} + C_2 \cos\frac{\pi y}{2l} + C_3 \cos\frac{\pi x}{2b}\cos\frac{\pi y}{2l}, \qquad (10.3)$$

where C_0, C_1, C_2, C_3 are constants having the dimension of length.

The first term in (10.3) determines the translational displacement of the entire plate, the second and third terms represent the cylindrical bending of the plate in the x and y directions respectively, while the fourth term defines the three-dimensional bending.

The coefficients C_i in (10.3) are determined by Bubnov and Galerkin's variational method based on the equilibrium conditions, i.e., equating the total work done by all external and internal forces acting on the plate over each virtual displacement to zero:

$$\left.\begin{aligned} \bar{w}_0 &= 1, & \bar{w}_2 &= \cos\frac{\pi y}{2l}, \\ \bar{w}_1 &= \cos\frac{\pi x}{2b}, & \bar{w}_3 &= \cos\frac{\pi x}{2b}\cos\frac{\pi y}{2l}. \end{aligned}\right\} \qquad (10.4)$$

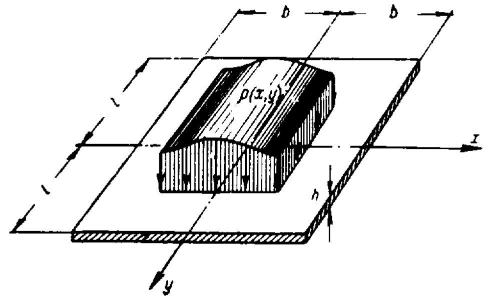


FIGURE 94.

2. Determining the reactions of the elastic foundation

An analysis of (10.1) shows that the first term depends on the internal forces in the plate, while the other terms depend on the reactions of the elastic foundation, distributed over the surface supporting the plate and caused by the compressive and shearing strains in the elastic foundation.

In addition to these forces and to the distributed load $p(x, y)$, reactions Q^{Φ}, distributed along its edges act on the plate. These reactions are introduced to make allowance for the three-dimensional deformation of the elastic foundation beyond the plate edges. In the case of rectangular or polygon-shaped plates, concentrated reactions R^{Φ} arise at the plate corners (Figure 95). In order to determine the reactions Q^{Φ} and R^{Φ} we shall assume that the vertical displacements w_f of the elastic-foundation surface beyond the plate edges obey the following law (Figure 96)*:

in the positive direction of the x axis

$$w_f(x, y) = w_l(y)\, e^{-\alpha(x-b)} \qquad (10.5)$$

in the positive direction of the x axis

$$w_f(x, y) = w_b(x)\, e^{-\alpha(y-l)}, \qquad (10.6)$$

where $\alpha = \sqrt{\frac{k}{2t}}$, $w_l(y)$ and $w_b(x)$ are respectively the vertical displacements of the longitudinal and lateral plate edges. The following law is also assumed for the vertical displacements of the foundation in the region $x > b$, $y > l$:

$$w_f(x, y) = w_c\, e^{-\alpha(x-b)}\, e^{-\alpha(y-l)}, \qquad (10.7)$$

where w_c is the vertical displacement of the plate corner.

It was shown in section 2 of this chapter that if the distribution of the vertical displacements of the foundation beyond the plate edges is given by (10.5), the fictitious reactions Q_l^{Φ} at the longitudinal plate edges will be given by (2.19), which can be written in the form:

$$Q_l^{\Phi} = 2t\left[\alpha w_l + \left(\frac{\partial w}{\partial x}\right)_l - \frac{1}{2\alpha}\left(\frac{\partial^2 w}{\partial y^2}\right)_l\right], \qquad (10.8)$$

* These expressions are only approximate, since in the three-dimensional problem the vertical displacements of the foundation beyond the plate edges obey a more complex law (see for instance section 7 of Chapter I).

where the derivatives of $w(x, y)$ are taken at $x = \pm b$.

The fictitous reactions Q_b^{Φ} distributed over the lateral plate edges are obtained similarly. Defining these reactions as the work done by all forces in a strip of unit width, cut from the elastic foundation, over the virtual displacement of the elastic foundation beyond the plate edges, we obtain:

$$Q_b^{\Phi} = 2t\left[\alpha w_b + \left(\frac{\partial w}{\partial y}\right)_b - \frac{1}{2\alpha}\left(\frac{\partial^2 w}{\partial x^2}\right)_b\right], \tag{10.9}$$

where the derivatives of $w(x, y)$ are taken at $y = \pm l$.

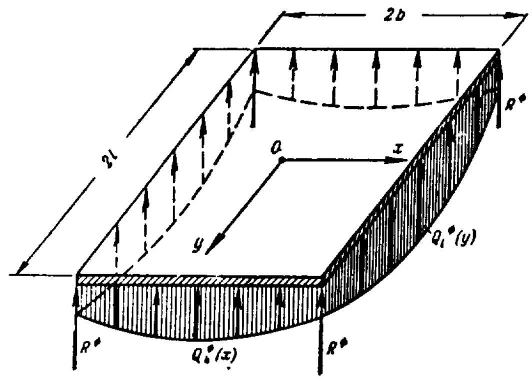


FIGURE 95.

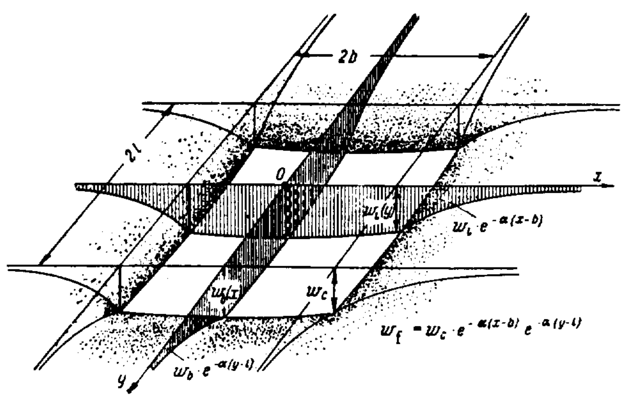


FIGURE 96.

The concentrated fictitious reactions R^{Φ} are determined by the vertical displacements of the elastic foundation beyond the plate edges in the regions:

$$(x \leqslant -b,\ y \leqslant -b),\ (x \leqslant -b,\ y \geqslant l),\ (x \geqslant b,\ y \leqslant -l),\ (x \geqslant b,\ y \geqslant l).$$

These reactions are defined as the work done by all the internal forces in the elastic foundation in the corresponding regions:

$$R^{\Phi} = \frac{3}{2} t w_c, \qquad (10.10)$$

where t is given by (2.15) and w_c is the vertical displacement of the corresponding plate corner.

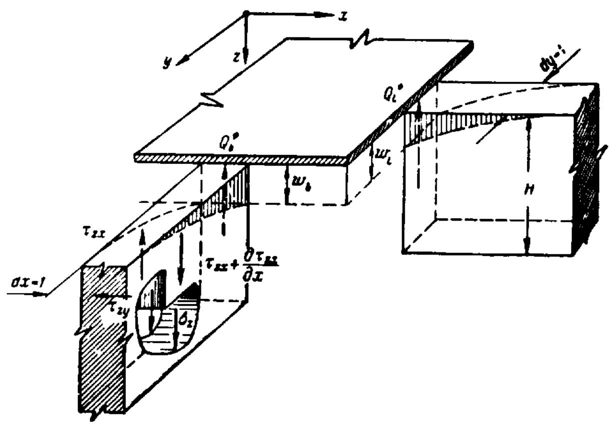


FIGURE 97.

Indeed, for $x \geqslant b$, [$y \geqslant l$], the vertical displacements $w_f(x, y)$ of the surface of the elastic foundation are given by (10.7). The virtual displacements of the elastic foundation are therefore:

$$\bar{w}(x, y, z) = \bar{w}_f(x, y)\psi(z) = 1 \cdot e^{-\alpha(x-b)} e^{-\alpha(y-l)} \psi(z). \text{ [when } w_c = 1 \text{]} \qquad (10.11)$$

The internal forces in the elastic foundation are the stresses σ_z, τ_{zx}, τ_{zy}, :

$$\left.\begin{aligned} \sigma_z &= \frac{E_0}{1-\nu_0^2}\psi'(z)\, w_f(x, y), \\ \tau_{zy} &= \frac{E_0}{2(1+\nu_0)}\psi(z)\frac{\partial w_f(x, y)}{\partial y}, \\ \tau_{zx} &= \frac{E_0}{2(1+\nu_0)}\psi(z)\frac{\partial w_f(x, y)}{\partial x}. \end{aligned}\right\} \qquad (10.12)$$

[cf. (2.13)]

The work done by these stresses over the virtual displacements (10.11) for $x \geqslant b$, $y \geqslant l$ is:

$$R^{\Phi} = \int_b^{\infty}\int_l^{\infty}\int_0^{H}\left(\sigma_z \bar{w}_f \psi' + \tau_{zx}\frac{\partial \bar{w}_f}{\partial x}\psi + \tau_{zy}\frac{\partial \bar{w}_f}{\partial y}\psi\right) dx\, dy\, dz, \text{ [cf. (2.11), (2.12)]} \qquad (10.13)$$

where

$$\bar{w}_f(x, y) = e^{-\alpha(x-b)} e^{-\alpha(y-l)}.$$

Substituting (10.12) in (10.13) and integrating we obtain (10.10).

The reactions R^{Φ} given by (10.10) are relatively small and their influence on the strains of the plate is insignificant. Furthermore, plastic deformation occurs in practice near the plate corners. The reactions R^{Φ} can therefore be neglected.

3. Variational equilibrium conditions

The following four algebraic equations in the four integration constants appearing in (10.3) are obtained by forming the expressions for the work done by all external and internal forces in the plate over the virtual displacements (10.4):

$$\left.\begin{aligned} &\iint [kw - p]\,dx\,dy + 2\int Q_b^{\Phi}\,dx + 2\int Q_l^{\Phi}\,dy + 4R^{\Phi} = 0,\\ &\iint\left[D\,\frac{\partial^4 w}{\partial x^4} - 2t\,\frac{\partial^2 w}{\partial x^2} + kw - p\right]\cos\frac{\pi x}{2b}\,dx\,dy +\\ &\qquad\qquad + 2\int Q_b\cos\frac{\pi x}{2b}\,dx = 0,\\ &\iint\left[D\,\frac{\partial^4 w}{\partial y^4} - 2t\,\frac{\partial^2 w}{\partial y^2} + kw - p\right]\cos\frac{\pi y}{2l}\,dx\,dy +\\ &\qquad\qquad + 2\int Q_l\cos\frac{\pi y}{2l}\,dy = 0,\\ &\iint\left[D\left(\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) - 2t\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}\right) +\right.\\ &\qquad\qquad \left. + kw - p\right]\cos\frac{\pi x}{2b}\cos\frac{\pi y}{2l}\,dx\,dy = 0, \end{aligned}\right\} \qquad (10.14)$$

where w is given by (10.3); $p = p(x, y)$ is the known external load and Q_l^{Φ}, Q_b^{Φ}, R^{Φ} are given by (10.8), (10.9), and (10.10) respectively. The integrals in (10.14) are definite and have the following limits: $-b \leqslant x \leqslant b$, $-l \leqslant y \leqslant l$. When concentrated external loads are present, these integrals are to be understood as Stieltjes integrals.

The first equation (10.14) defines the work done by all forces external with respect to the plate over the vertical displacement $\bar{w}_0 = 1$. In the term containing k allowance is made for the compressive strains in the elastic foundation.

The second equation (10.14) defines the work done by all forces during the cylindrical bending of the plate in the z plane. By the terms containing D and t allowance is made for the work done by the bending moments M_x, and by the shearing strains in the elastic foundation respectively.

Similarly, the third equation defines the work done by all forces during cylindrical bending of the plate in the yz plane. In this equation, the term containing D corresponds to the work done by the bending moments M_y.

The last equation (10.14) corresponds to three-dimensional bending of the plate, similar to the bending of a plate simply supported along the edges. In this case, the work done by the internal forces consists of the work done by the bending moments M_x and M_y, and the twisting moments H.

Substitution of (10.3), (10.8), (10.9), and (10.10) in (10.14) yields the system of four algebraic equations (10.15) from which the coefficients

C_0, C_1, C_2, C_3 can be obtained:

TABLE 11

C_0	C_1	C_2	C_3	Free term	Virtual displacements
k_{00}	k_{10}	k_{20}	k_{30}	G_0	$\overline{w}_0$ 
•	n_{11}	n_{21}	n_{31}	G_1	$\overline{w}_1$ 
•	•	n_{22}	n_{32}	G_2	$\overline{w}_2$ 
•	•	•	n_{33}	G_3	$\overline{w}_3$ 

(10.15)

Here

$$\left.\begin{aligned}
k_{00} &= 4\left[lbk + 2\alpha t\,(l+b) + \frac{3}{2}t\right],\\
k_{10} &= \frac{8}{\pi}\,[lbk + 2\alpha tb],\\
k_{20} &= \frac{8}{\pi}\,[lbk + 2\alpha tl],\\
k_{30} &= \frac{16}{\pi^2}\,lbk;
\end{aligned}\right\} \qquad (10.16)$$

$$\left.\begin{aligned}
n_{11} &= 2\left[lbk + \frac{\pi^2}{2}\frac{l}{b}t + \frac{\pi^4}{16}\frac{l}{b^3}D + 2\alpha tb + \frac{\pi^2}{4}\frac{t}{ab}\right],\\
n_{21} &= \frac{16}{\pi^2}\,lbk,\\
n_{31} &= \frac{4}{\pi}\left[lbk + \frac{\pi^2}{2}\frac{l}{b}t + \frac{\pi^4}{16}\frac{l}{b^3}D\right],\\
n_{22} &= 2\left[lbk + \frac{\pi^2}{2}\frac{b}{l}t + \frac{\pi^4}{16}\frac{b}{l^3}D + 2\alpha tl + \frac{\pi^2}{4}\frac{t}{al}\right],\\
n_{32} &= \frac{4}{\pi}\left[lbk + \frac{\pi^2}{2}\frac{b}{l}t + \frac{\pi^4}{16}\frac{b}{l^3}D\right],\\
n_{33} &= lbk + \frac{\pi^2}{2}\left(\frac{l}{b} + \frac{b}{l}\right)t + \frac{\pi^4}{16}D\left(\frac{l}{b^3} + \frac{2}{lb} + \frac{b}{l^3}\right),
\end{aligned}\right\} \qquad (10.17)$$

and

$$D = \frac{Eh^3}{12\,(1-\mu^2)}$$

is the flexural rigidity of the plate, while k and t are given by (10.2).

It is seen that the coefficients (10.16) determine the work done by the reactions of the elastic foundation, while the coefficients (10.17) determine the work done by the reactions and the internal forces in the plate. These coefficients are symmetrical:

$$\left.\begin{array}{lll} k_{10}=k_{01}, & k_{20}=k_{02}, & k_{30}=k_{03}, \\ n_{12}=n_{21}, & n_{13}=n_{31}, & n_{23}=n_{32}, \end{array}\right\} \tag{10.18}$$

in accordance with Maxwell and Betti's reciprocity theorem.

The matrix of (10.15) is symmetrical by virtue of (10.18), which considerably simplifies the determination of the unknown constants when more than four terms are taken in (10.3).

The free terms on the right sides of (10.15) (Table 11) represent the work done by the known external load over the corresponding displacements (10.4), and are obtained in the form:

$$G_i = \iint p(x,\ y)\,\bar{w}_i\,dx\,dy. \tag{10.19}$$

4.

System (10.15) could have been obtained without introducing the fictitious reactions Q_l^{Φ}, Q_b^{Φ}, R^{Φ}, had we followed the procedure in section 2 in deriving the generalized equilibrium conditions of an elementary strip of width dy, in which we considered the work done by all internal forces, including those acting in the elastic foundation beyond the plate edges. The final result would have been the same, since the fictitious forces Q_l^{Φ}, Q_b^{Φ}, R^{Φ} were defined as the work done by all the internal forces acting in the elastic foundation beyond the plate edges, and were only introduced to simplify the expressions.

5

After the constants C_0, C_1, C_2, C_3 have been determined from (10.15), the plate deflections can be obtained from (10.3), the bending moments and shearing forces being given by (1.8). Substitution of (10.3) in (1.8) yields:

$$\left.\begin{aligned} M_x &= D\frac{\pi^2}{4b^2}\Big[\mu\frac{b^2}{l^2}C_2\cos\frac{\pi y}{2l} + C_1\cos\frac{\pi x}{2b} + \\ &\quad + \Big(1+\mu\frac{b^2}{l^2}\Big)C_3\cos\frac{\pi x}{2b}\cos\frac{\pi y}{2l}\Big], \\ M_y &= D\frac{\pi^2}{4b^2}\Big[\frac{b^2}{l^2}C_2\cos\frac{\pi y}{2l} + \mu C_1\cos\frac{\pi x}{2b} + \\ &\quad + \Big(\frac{b^2}{l^2}+\mu\Big)C_3\cos\frac{\pi x}{2b}\cos\frac{\pi y}{2l}\Big]. \end{aligned}\right\} \tag{10.20}$$

From (10.15) we obtain as particular cases approximate solutions for a plate simply supported along its entire contour, or only along either its lateral or its longitudinal edges. In this case we have to substitute $C_0 = C_1 = C_2 = 0$, $C_0 = C_1 = 0$, or $C_0 = C_2 = 0$ respectively in (10.3).

If a higher accuracy is desired, or if the external load has a pronounced nonuniform distribution, a larger number of terms must be taken in (10.3) (cf. section 12). We must then set up a system of algebraic equations similar to (10.15), each equation of which defines the work done by all external and internal forces acting on the plate over the corresponding displacements.

§ 11. EXAMPLES

1

Consider a rectangular plate, for which we assume that:

$$\psi(z) = \frac{\operatorname{sh}\gamma\frac{H-z}{b}}{\operatorname{sh}\frac{\gamma H}{b}} \tag{11.1}$$

where γ = coefficient depending on elastic properties of foundation; b = plate half-width.

The generalized characteristics of the elastic foundation are in this case: [cf. (5.23) and (5.24) of Chapter II]

$$\left.\begin{aligned} k &= \frac{E_0\gamma}{2b(1-\nu_0^2)} \cdot \frac{\operatorname{sh}\frac{\gamma H}{b}\operatorname{ch}\frac{\gamma H}{b}+\frac{\gamma H}{b}}{\operatorname{sh}^2\frac{\gamma H}{b}},\\ t &= \frac{E_0 b}{8\gamma(1+\nu_0)} \cdot \frac{\operatorname{sh}\frac{\gamma H}{b}\operatorname{ch}\frac{\gamma H}{b}-\frac{\gamma H}{b}}{\operatorname{sh}^2\frac{\gamma H}{b}},\\ \alpha &= \frac{\gamma}{b}\sqrt{\frac{2}{1-\nu_0}\,\frac{\operatorname{sh}\frac{\gamma H}{b}\operatorname{ch}\frac{\gamma H}{b}+\frac{\gamma H}{b}}{\operatorname{sh}\frac{\gamma H}{b}\operatorname{ch}\frac{\gamma H}{b}-\frac{\gamma H}{b}}}, \end{aligned}\right\} \tag{11.2}$$

where

$$\left.\begin{aligned} E_0 &= \frac{E_s}{1-\nu_s^2},\\ \nu_0 &= \frac{\nu_s}{1-\nu_s} \end{aligned}\right\} \tag{11.3}$$

(E_s and ν_s are, as before, the modulus of elasticity and Poisson's ratio for the elastic foundation respectively).

Substituting (11.2) in (10.16) and (10.17), and multiplying (10.15) by $\frac{1-\nu_0^2}{E_0 b}$, we obtain:

$$
\left.
\begin{aligned}
k_{00} &= 4\left[\frac{\gamma}{2}\frac{l}{b}m_k + \frac{1}{4}\left(1+\frac{l}{b}\right)\sqrt{6(1-\nu_0)}\,m_t m_a + \right.\\
&\qquad\qquad\qquad\qquad \left. + \frac{3}{16}\frac{1-\nu_0}{\gamma}m_t\right],\\
k_{10} &= \frac{8}{\pi}\left[\frac{\gamma}{2}\frac{l}{b}m_k + \frac{1}{4}\sqrt{6(1-\nu_0)}\,m_t m_a\right],\\
k_{20} &= \frac{8}{\pi}\left[\frac{\gamma}{2}\frac{l}{b}m_k + \frac{1}{4}\frac{l}{b}\sqrt{6(1-\nu_0)}\,m_t m_a\right],\\
k_{30} &= \frac{16}{\pi^2}\frac{\gamma}{2}\frac{l}{b}m_k;
\end{aligned}
\right\}
\qquad (11.4)
$$

$$
\left.
\begin{aligned}
n_{11} &= 2\left[\frac{\gamma}{2}\frac{l}{b}m_k + \frac{\pi^2}{16}\frac{1-\nu_0}{\gamma}\frac{l}{b}m_t + \frac{\pi^4}{16r}\left(\frac{l}{b}\right)^3 + \right.\\
&\qquad \left. + \frac{1}{4}\sqrt{6(1-\nu_0)}\,m_t m_a + \frac{\pi^2}{32}\frac{(1-\nu_0)^2}{\gamma^2\sqrt{6(1-\nu_0)}}\frac{m_t}{m_a}\right],\\
n_{21} &= \frac{16}{\pi^2}\frac{\gamma}{2}\frac{l}{b}m_k,\\
n_{31} &= \frac{4}{\pi}\left[\frac{\gamma}{2}\frac{l}{b}m_k + \frac{\pi^2}{16}\frac{1-\nu_0}{\gamma}\frac{l}{b}m_t + \frac{\pi^5}{16r}\left(\frac{l}{b}\right)^3\right],\\
n_{22} &= 2\left[\frac{\gamma}{2}\frac{l}{b}m_k + \frac{\pi^2}{16}\frac{1-\nu_0}{\gamma}\frac{b}{l}m_t + \frac{\pi^4}{16r}\frac{b}{l} + \right.\\
&\qquad \left. + \frac{1}{4}\sqrt{6(1-\nu_0)}\frac{l}{b}m_t m_a + \frac{\pi^2}{32}\frac{(1-\nu_0)^2}{\gamma^2\sqrt{6(1-\nu_0)}}\frac{m_t}{m_a}\frac{b}{l}\right],\\
n_{32} &= \frac{4}{\pi}\left[\frac{\gamma}{2}\frac{l}{b}m_k + \frac{\pi^2}{16}\frac{1-\nu_0}{\gamma}\frac{b}{l}m_t + \frac{\pi^5}{16r}\frac{b}{l}\right],\\
n_{33} &= \left[\frac{\gamma}{2}\frac{l}{b}m_k + \frac{\pi^2}{16}\frac{1-\nu_0}{\gamma}\left(1+\frac{l^2}{b^2}\right)\frac{b}{l} + \frac{\pi^4}{16r}\left(1+\frac{l^2}{b^2}\right)^2\frac{b}{l}\right],
\end{aligned}
\right\}
\qquad (11.5)
$$

where

$$
\left.
\begin{aligned}
m_k &= \frac{\operatorname{sh}\frac{\gamma H}{b}\operatorname{ch}\frac{\gamma H}{b} + \frac{\gamma H}{b}}{\operatorname{sh}^2\frac{\gamma H}{b}},\\
m_t &= \frac{\operatorname{sh}\frac{\gamma H}{b}\operatorname{ch}\frac{\gamma H}{b} - \frac{\gamma H}{b}}{\operatorname{sh}^2\frac{\gamma H}{b}},\\
m_a &= \sqrt{\frac{1}{3}\frac{\operatorname{sh}\frac{\gamma H}{b}\operatorname{ch}\frac{\gamma H}{b} + \frac{\gamma H}{b}}{\operatorname{sh}\frac{\gamma H}{b}\operatorname{ch}\frac{\gamma H}{b} - \frac{\gamma H}{b}}}.
\end{aligned}
\right\}
\qquad (11.6)
$$

and

$$
r = \frac{\pi E_0 l^3 b}{D(1-\nu_0^2)} \qquad (11.7)
$$

is the "flexibility index" of the plate. *

* A similar value for r is used by Gorbunov-Posadov /25, 26/.

The free terms in (10.15) are by (10.19):

$$\left.\begin{aligned}
G_1 &= \frac{1-\nu_0^2}{E_0 b}\iint p(x,\ y)\,dx\,dy,\\
G_2 &= \frac{1-\nu_0^2}{E_0 b}\iint p(x,\ y)\ \cos\frac{\pi x}{2b}\,dx\,dy,\\
G_3 &= \frac{1-\nu_0^2}{E_0 b}\iint p(x, y)\ \ \cos\frac{\pi y}{2l}\,dx\,dy,\\
G_4 &= \frac{1-\nu_0^2}{E_0 b}\iint p(x,\ y)\cos\frac{\pi x}{2b}\cos\frac{\pi y}{2l}\,dx\,dy.
\end{aligned}\right\} \qquad (11.8)$$

2. Approximative analysis for a uniformly distributed load

Consider a rectangular plate acted upon by a uniformly distributed load of intensity p (Figure 98). We assume that:

$$l = 2b, \qquad \gamma = 1.5, \qquad r = 1.0, \qquad \nu_0 = 0.4. \qquad (11.9)$$

and that the plate lies on an elastic foundation of infinite thickness, $H = \infty$.

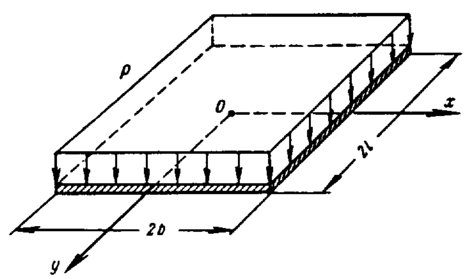


FIGURE 98.

Substitution of (11.9) in (11.4) and (11.5) yields:

$$\left.\begin{aligned}
&k_{00} = 9.58, && n_{11} = 310.4, && n_{22} = 23.50,\\
&k_{10} = 4.51, && n_{21} = 2.44, && n_{32} = 14.22,\\
&k_{20} = 5.23, && n_{31} = 197.8; && n_{33} = 241.1.\\
&k_{30} = 2.44;
\end{aligned}\right\} \qquad (11.10)$$

The load terms (11.8) become for $p = \text{const}$:

$$\left.\begin{aligned}
G_0 &= 4\,\frac{1-\nu_0^2}{E_0}\,pl,\\
G_1 &= \frac{8}{\pi}\,\frac{1-\nu_0^2}{E_0}\,pl,\\
G_2 &= \frac{8}{\pi}\,\frac{1-\nu_0^2}{E_0}\,pl,\\
G_3 &= \frac{16}{\pi^2}\,\frac{1-\nu_0^2}{E_0}\,pl.
\end{aligned}\right\} \qquad (11.11)$$

The following formulas are obtained for C_0, C_1, C_2, C_3 by substituting (11.10) and (11.11) in (10.15) and solving:

$$\left.\begin{aligned} C_0 &= 408\cdot 10^{-3}\frac{1-\nu_0^2}{E_0}pl, \\ C_1 &= 2{,}37\cdot 10^{-3}\frac{1-\nu_0^2}{E_0}pl, \\ C_2 &= 17{,}4\cdot 10^{-3}\frac{1-\nu_0^2}{E_0}pl, \\ C_3 &= 0{,}4\cdot 10^{-3}\,\frac{1-\nu_0^2}{E_0}pl. \end{aligned}\right\} \qquad (11.12)$$

Hence,

$$w(x,y) = \left[408 + 2{,}37\cos\frac{\pi x}{2b} + 17{,}4\cos\frac{\pi y}{2l} - \right.$$
$$\left. - 0{,}4\cos\frac{\pi x}{2l}\cos\frac{\pi y}{2l}\right]\frac{pl\,(1-\nu_0^2)}{E_0}\cdot 10^{-3}. \qquad (11.13)$$

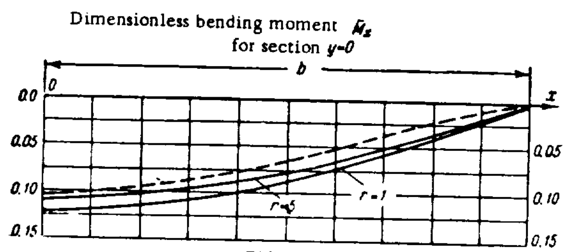


FIGURE 99.

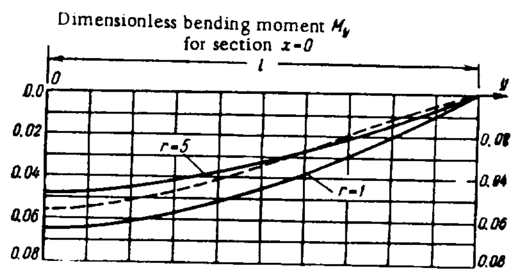


FIGURE 100.

Figures 99 and 100 show the dimensionless bending moments $\overline{M}_x$ and $\overline{M}_y$, at $y=0$ and $x=0$ respectively, determined from (10.20) and (11.12) for $\mu=0$. The actual bending moments are:

$$M_x = \overline{M}_x pb^2, \quad M_y = \overline{M}_y pl^2.$$

Bending moments, obtained for $r=5$ by this method, as well as by Gorbunov-Posadov's method for a rigid plate (broken line) are also shown.*

* See Gorbunov-Posadov, M. I. Raschet konstruktsii na unrugom osnovanii (Analyzing Structures on Elastic Foundations), p. 457. 1953.

In view of the approximative character of our method, and taking into account that according to Gorbunov-Posadov, plates are rigid when

$$r < \frac{8}{\sqrt{\frac{l}{b}}},$$

the agreement between the results obtained by the two methods can be considered as satisfactory.

3. Approximative analysis of the foundation slab of a spillway dam

Figure 101 shows a section of a spillway dam of light-weight design. The foundation of this section is a rectangular concrete slab of constant rigidity having piers at its lateral edges. One of the most critical stages is the period when the slab and piers have already been erected, but the spillway sections are not yet in place. This case will now be considered.

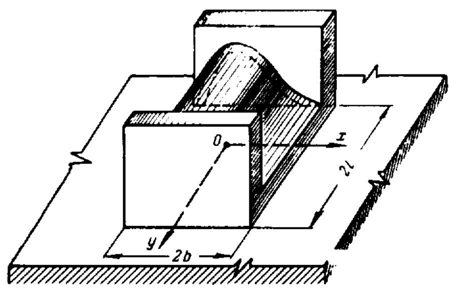
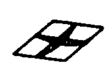


FIGURE 101.

Since the rigidity of the piers in their planes is very large, the lateral plate edges can be considered as unbendable. The joint between the slab and the piers can be considered as a hinged support. Applying (10.3) to the slab deflections, we must put $C_1 = 0$. We then obtain Table 12 from (10.15).

TABLE 12

C_0	C_2	C_3	Free term	Virtual displacements
k_{00}	k_{22}	k_{30}	G_0	$\bar{w}_0$ 
•	n_{22}	n_{32}	G_2	$\bar{w}_2$ 
•	•	n_{33}	G_3	$\bar{w}_3$ 

The coefficients in Table 12 are given, as before, by (11.4), (11.5). Assuming that a load p is uniformly distributed over the entire slab and a load g is uniformly distributed along the lateral edges (Figure 102), we obtain:

$$\left.\begin{aligned} G_0 &= 4(pl+g)\frac{1-\nu_0^2}{E_0}, \\ G_2 &= \frac{8}{\pi}pl\frac{1-\nu_0^2}{E_0}, \\ G_3 &= \frac{16}{\pi^2}pl\frac{1-\nu_0^2}{E_0}. \end{aligned}\right\} \tag{11.14}$$

After the constants C_0, C_2, C_3 have been determined from the equations in Table 12, we obtain for the slab deflections:

$$w(x,\ y) = C_0 + C_2\cos\frac{\pi y}{2l} + C_3\cos\frac{\pi x}{2b}\cos\frac{\pi y}{2l}; \tag{11.15}$$

The bending moments are therefore by (10.20):

$$\left.\begin{aligned} M_x &= D\frac{\pi^2}{4b^2}\left[\mu\frac{b^2}{l^2}C_2\cos\frac{\pi y}{2l} + \left(1+\mu\frac{b^2}{l^2}\right)C_3\cos\frac{\pi x}{2b}\cos\frac{\pi y}{2l}\right], \\ M_y &= D\frac{\pi^2}{4b^2}\left[\frac{b^2}{l^2}C_2\cos\frac{\pi y}{2l} + \left(\frac{b^2}{l^2}+\mu\right)C_3\cos\frac{\pi x}{2b}\cos\frac{\pi y}{2l}\right]. \end{aligned}\right\} \tag{11.16}$$

4. Taking into account additional loads transmitted by the adjacent sections

Consider now the case when a system of slabs, loaded symmetrically and arranged in a row, lies on a soil foundation. We thus consider not a separate section, but the dam as a whole (Figure 103, a). If the base of each section is an absolutely rigid plate, no shearing forces will act in the elastic foundation at the boundaries between the different sections. The concentrated reactions Q_b^0 at the lateral plate edges will therefore vanish, as can be seen from (10.9).

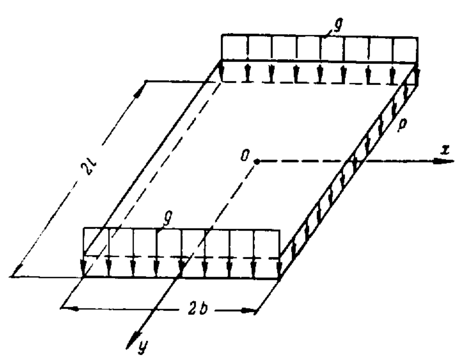
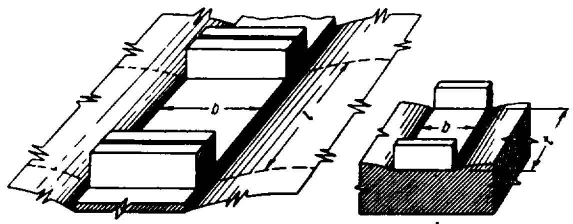


FIGURE 102.

If each plate lying on the elastic foundation has a finite but sufficiently large rigidity, the shearing forces acting in the elastic foundation at the boundaries of the different sections will be small, so that the concentrated reactions Q_b^{Φ} may be neglected. In this case we shall be able to disregard the influence exerted by the adjacent sections, and analyze such plates according to the scheme in Figure 103, b.



a b

FIGURE 103.

By expressing the deflection of each plate in the form (10.3), we obtain, as before, the system of algebraic equations (10.15) (Table 11). Since, however, Q_b^{Φ} and R^{Φ} are zero, the coefficients of this system are:

$$\left.\begin{aligned}
k_{00} &= 4\left[\frac{\gamma}{2}\frac{l}{b}m_k + \frac{1}{4}\frac{l}{b}\sqrt{6(1-\nu_0)}m_t m_a\right],\\
k_{10} &= \frac{8}{\pi}\left[\frac{\gamma}{2}\frac{l}{b}m_k\right],\\
k_{20} &= \frac{8}{\pi}\left[\frac{\gamma}{2}\frac{l}{b}m_k + \frac{1}{4}\frac{l}{b}\sqrt{6(1-\nu_0)}m_t m_a\right],\\
k_{30} &= \frac{16}{\pi^2}\frac{\gamma}{2}\frac{l}{b}m_k;
\end{aligned}\right\} \tag{11.17}$$

$$\left.\begin{aligned}
n_{11} &= 2\left[\frac{\gamma}{2}\frac{l}{b}m_k + \frac{\pi^2}{16}\frac{1-\nu_0}{\gamma}\frac{l}{b}m_t + \frac{\pi^5}{16r}\left(\frac{l}{b}\right)^3\right],\\
n_{21} &= \frac{16}{\pi^2}\frac{\gamma}{2}\frac{l}{b}m_k,\\
n_{31} &= \frac{4}{\pi}\left[\frac{\gamma}{2}\frac{l}{b}m_k + \frac{\pi^2}{16}\frac{1-\nu_0}{\gamma}\frac{l}{b}m_t + \frac{\pi^5}{16r}\left(\frac{l}{b}\right)^3\right],\\
n_{22} &= 2\left[\frac{\gamma}{2}\frac{l}{b}m_k + \frac{\pi^2}{16}\frac{1-\nu_0}{\gamma}\frac{b}{l}m_t + \frac{\pi^5}{16r}\frac{b}{l} + \right.\\
&\qquad \left. + \frac{1}{4}\sqrt{6(1-\nu_0)}\frac{l}{b}m_t m_a + \frac{\pi^2}{32}\frac{1-\nu_0^2}{\gamma^2\sqrt{6(1-\nu_0)}}\frac{m_t}{m_a}\frac{b}{l}\right],\\
n_{32} &= \frac{4}{\pi}\left[\frac{\gamma}{2}\frac{l}{b}m_k + \frac{\pi^2}{16}\frac{1-\nu_0}{\gamma}\frac{b}{l}m_t + \frac{\pi^5}{16r}\frac{b}{l}\right],\\
n_{33} &= \left[\frac{\gamma}{2}\frac{l}{b}m_k + \frac{\pi^2}{16}\frac{1-\nu_0}{\gamma}m_t\left(1+\frac{l^2}{b^2}\right)\frac{b}{l} + \right.\\
&\qquad \left. + \frac{\pi^5}{16r}\left(1+\frac{l^2}{b^2}\right)\frac{b}{l}\right],
\end{aligned}\right\} \tag{11.18}$$

where m_k, m_t, m_a and r are given by (11.6) and (11.7).

The free terms are, as before:

$$G_i = \frac{1-\nu_0^2}{E_0 b}\iint p(x,\ y)\,\bar{w}_i\,dx\,dy. \tag{11.19}$$

If we assume that the lateral plate edges do not bend, due to the presence of perfectly rigid piers, the deflections will be given by (11.15). The constants C_0, C_2, C_3 are determined, as before, by the system of algebraic equations in Table 12, the coefficients being given by (11.17) and (11.18).

By making allowance for the additional loads transmitted from the adjacent sections, marked reduction of the positive, and an increase in the negative, bending moments M_y may result, the general deformation pattern changing considerably.

§ 12. GENERAL CASE OF LOADING OF A PLATE HAVING FREE EDGES

1. Method of solution

Consider the general case of loading of a rectangular plate lying freely on an elastic foundation. Let the external load consist of concentrated vertical forces P and of forces $p(x, y)$ distributed over the plate.

To solve this problem we have to find the deflections $w(x, y)$ from (10.1) for given boundary conditions. When the plate edges are neither built-in nor loaded, the statical boundary conditions are [by (1.8) and (1.9)]:

at $x = \pm\frac{b}{2}$

$$\left.\begin{aligned} M_x &= -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) = 0, \\ Q_x &= -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right] = Q_l^\Phi, \end{aligned}\right\} \tag{12.1}$$

at $y = \pm\frac{l}{2}$

$$\left.\begin{aligned} M_y &= -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) = 0, \\ Q_y &= -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2\,\partial y}\right] = Q_b^\Phi, \end{aligned}\right\} \tag{12.2}$$

where Q_l^Φ, Q_b^Φ, determining the strain of the elastic foundation beyond the plate edges, are given by (10.8) and (10.9) respectively.

The problem will be solved by Bubnov and Galerkin's variational method, in which the deflection function $w(x, y)$ is represented as a series each term of which satisfies the boundary conditions:

$$w(x, y) = \sum_1^m \sum_1^n C_{mn}\varphi_{mn}(x, y), \tag{12.3}$$

where $\varphi_{mn}(x, y)$ are known functions, and C_{mn} are constants which have to be determined.

The functions φ_{mn} can be selected arbitrarily, provided they are linearly independent and satisfy the geometrical boundary conditions of the problem. Rigorous fulfilment of the statical boundary conditions is not required, since, when setting up the Lagrange equations, the equilibrium conditions are approximately satisfied at all points of the plate.

We represent the functions φ_{mn} as trigonometric functions, together with linear terms defining the translational displacements of the entire plate. We thus satisfy the geometrical boundary conditions, since at the plate edges $w \neq 0$, $\frac{\partial w}{\partial n} \neq 0$, and also, for $\mu = 0$, the first of each statical condition (12.1) and (12.2). The remaining statical conditions are fulfilled only approximately.

Proceeding from Lagrange's principle of virtual displacements, we can establish a system of algebraic equations for determining the constant coefficients C_{mn} in (12.3); in each equation the work done by all external and internal forces acting on the plate over the virtual displacement φ_{ik} is equated to zero:

$$\sum_{1}^{m}\sum_{1}^{n} C_{mn}\left\{\iint [D\nabla^2\nabla^2\varphi_{mn} - 2t\nabla^2\varphi_{mn} + k\varphi_{mn} - p]\,\varphi_{ik}\,dx\,dy + \right.$$
$$\left. + \oint [Q_{mn}(s) + Q^{\Phi}(s)]\,\varphi_{ik}(s)\,ds\right\} = 0 \qquad (12.4)$$
$$(i = 1, 2, 3, \ldots, m; \qquad k = 1, 2, 3, \ldots, n),$$

where $\varphi_{mn}(s)$, $\varphi_{ik}(s)$ are the values of the corresponding functions at the contour.

The double integral in (12.4) defines the work done by the internal forces acting in the plate (bending and twisting moments), the work done by the shearing and compressive stresses in the elastic foundation beneath the plate, and the work done by the external load. The contour integral defines the work done by the shearing forces acting on the plate edges over their virtual displacements. The first term represents the work done by Kirchhoff's reduced shearing forces (cf. (1.9)), which appear at the plate edges because the static-equilibrium conditions (12.1), (12.2) are only approximately satisfied. The second term represents the work done by the reactions (10.8), (10.9), acting at the plate edges and determining the deformation of the elastic foundation beyond them.

As already stated, the concentrated reactions R^{Φ} acting at the plate corners (given by (10.10)) can in practice be neglected, so that the work done by them is not taken into account in (12.4).

The integrals in (12.4) are taken over the entire area and the entire contour of the plate respectively. In the presence of concentrated external loads, these integrals are to be understood as Stieltjes integrals. Thus, for a finite number of concentrated forces, the integrals should be replaced by the sums of the products of each force by the function φ_{ik} at its point of action.

We can rewrite (12.4) in canonical form:

$$\left.\begin{array}{l}
\delta_{00,00}C_{00} + \delta_{00,10}C_{10} + \ldots + \delta_{00,mn}C_{mn} = \Delta_{00},\\
\delta_{10,00}C_{00} + \delta_{10,10}C_{10} + \ldots + \delta_{10,mn}C_{mn} = \Delta_{10},\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\delta_{ik,00}C_{00} + \delta_{ik,10}C_{10} + \ldots + \delta_{ik,mn}C_{mn} = \Delta_{ik},\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\delta_{mn,00}C_{00} + \delta_{mn,10}C_{10} + \ldots + \delta_{mn,mn}C_{mn} = \Delta_{mn}.
\end{array}\right\} \qquad (12.5)$$

where

$$\delta_{ik,mn} = \iint [D\nabla^2\nabla^2\varphi_{mn} - 2t\nabla^2\varphi_{mn} + k\varphi_{mn}]\,\varphi_{ik}\,dx\,dy + \\ + \oint [Q_{mn}(s) + Q^{\Phi}(s)]\,\varphi_{ik}(s)\,ds. \qquad (12.6)$$

These coefficients are symmetrical ($\delta_{ik,mn} = \delta_{mn,ik}$) by virtue of Maxwell and Betti's reciprocity theorem; the matrix of (12.5) is therefore symmetrical.

The free terms in (12.5) are:

$$\Delta_{ik} = \iint p(x,\ y)\,\varphi_{ik}\,dx\,dy \qquad (12.7)$$

and represent the work done by the external load over each virtual displacement.

In the solution of practical problems it is convenient to resolve the external load into four symmetrical and antisymmetrical components. For example, Figure 104 represents the resolution of a concentrated force applied at $x = a,\ y = c$.

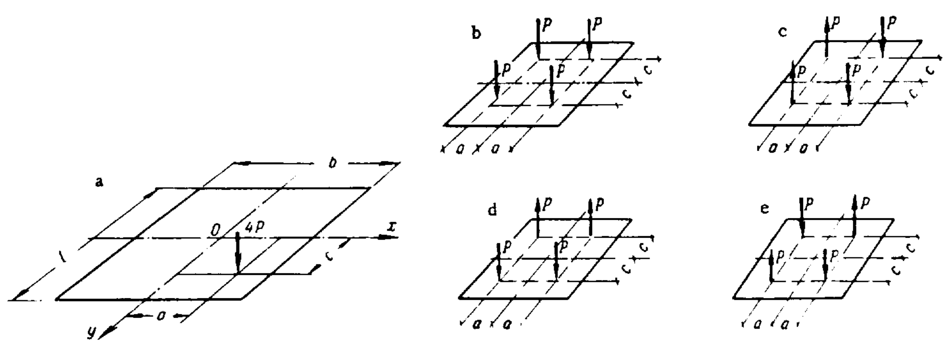


FIGURE 104.

The calculations are considerably reduced when each load component is analyzed separately.

2. Symmetrical load

When the load is symmetrical with respect to both axes (Figure 104, b), (12.3) becomes:

$$w(x,\ y) = C_{00} + \sum_{1}^{m} C_{m0}\cos\frac{m\pi x}{b} + \sum_{1}^{n} C_{0n}\cos\frac{n\pi y}{l} + \sum_{1}^{m}\sum_{1}^{n} C_{mn}\cos\frac{m\pi x}{b}\cos\frac{n\pi y}{l} \qquad (12.8)$$

$$(m, n = 1, 3, 5, 7, \ldots, (2k-1)).$$

By forming the expressions for the work done by all forces acting on the plate over each virtual displacement, we obtain the coefficients (12.6). When these are inserted into (12.5), we can solve this system for the unknown constants C_{00}, C_{m0}, C_{0n}, C_{mn}.

Table 13 presents the matrix of the algebraic equations when nine terms ($m=3$, $n=3$) are taken in (12.8), corresponding to nine possible displacements of the plate: translational displacement of the entire plate, four displacements characterizing cylindrical bending in the xz and yz planes respectively, and four displacements similar to the deflection of a plate simply supported along the contour.

It can be seen that the matrix is symmetrical about the principal diagonal. It is therefore necessary to obtain 29 dimensionless coefficients.

The magnitudes α, β, k, t and D entering in these coefficients are determined by the formulas:

$$\left.\begin{aligned} &\alpha=\sqrt{\frac{k}{2t}},\quad \beta=\frac{b}{l},\\ &k=\frac{E_0}{1-\nu_0^2}\int_0^H \phi'^2\,dz,\\ &t=\frac{E_0}{4(1+\nu_0)}\int_0^H \phi^2\,dz,\\ &D=\frac{Eh^3}{12(1-\mu^2)}.\end{aligned}\right\} \tag{12.9}$$

Here l and b = length and width of plate respectively; D = flexural rigidity of plate; k and t = generalized characteristics of elastic foundation; $\phi=\phi(z)$ = = funotion describing the distribution of displacements over the depth of the elastic foundation;

and
$$E_0=\frac{E_s}{1-\nu_s^2},\qquad \nu_0=\frac{\nu_s}{1-\nu_s}. \tag{12.10}$$

The free terms are:

$$A_{ik}=\frac{1}{klb}\iint p(x,\ y)\,\varphi_{ik}\,dx\,dy. \tag{12.11}$$

When concentrated loads are present, the integrals in (12.11) are to be understood as Stieltjes integrals.

3. Load symmetrical with respect to one, and antisymmetrical with respect to the other axis

If the external load is symmetrical with respect to the x axis and antisymmetrical with respect to the y axis (Figure 104, c), we can write:

$$w(x,\ y)=C_{00}\frac{2x}{b}+\sum_2^m C_{m0}\sin\frac{m\pi x}{b}+\frac{2x}{b}\sum_1^n C_{0n}\cos\frac{n\pi y}{l}+ \\ +\sum_2^m\sum_1^n C_{mn}\sin\frac{m\pi x}{b}\cos\frac{n\pi y}{l} \tag{12.12}$$

$$(m=2,\ 4,\ 6,\ldots;\qquad n=1,\ 3,\ 5,\ldots).$$

The first term defines the rotation of the entire plate about the y axis. The single series defines the cylindrical bending in the xz plane, and the deformation of the longitudinal plate edges respectively. The double series defines the displacements corresponding to a plate, simply supported along its contour. Inserting (12.12) into (12.6) all the coefficients of the solving system (12.5) can be found. Exactly as for symmetrical loading, we restrict ourselves to nine terms in (12.12) ($m=2; 4; n=1; 3$). The system of nine algebraic equations thus obtained is presented in Table 14. The free terms in these equations represent the work done by the known external load over the corresponding virtual displacements and are obtained from (12.11). The magnitudes α, β, k and D are given by (12.9).

In the similar case of a load, symmetrical with respect to the y axis and antisymmetrical with respect to the x axis (Figure 104, d), we can write:

$$w(x, y) = C_{00}\frac{2y}{l} + \frac{2y}{l}\sum_{1}^{m} C_{m0}\cos\frac{m\pi x}{b} + \\ + \sum_{2}^{n} C_{0n}\sin\frac{n\pi y}{l} + \sum_{1}^{m}\sum_{2}^{n} C_{mn}\cos\frac{m\pi x}{b}\sin\frac{n\pi y}{l} \tag{12.13}$$

$$(m = 1,\ 3,\ 5, \ldots; \qquad n = 2,\ 4,\ 6, \ldots).$$

The matrix of the algebraic equations for this case ($m=1; 3; n=2; 4$) is represented in Table 15.

4. Antisymmetrical load

When the load is antisymmetrical with respect to both axes (Figure 104,e), we can write:

$$w(x, y) = C_{00}\frac{4xy}{lb} + \frac{2y}{l}\sum_{2}^{m} C_{m0}\sin\frac{m\pi x}{b} + \\ \frac{2x}{b}\sum_{2}^{n} C_{0n}\sin\frac{n\pi y}{l} + \sum_{2}^{m}\sum_{2}^{n} C_{mn}\sin\frac{m\pi x}{b}\sin\frac{n\pi y}{l} \tag{12.14}$$

$$(m,\ n = 2,\ 4,\ 6,\ 8, \ldots).$$

The first term defines the deformation of the entire plate, in which the edges remain straight. The single series defines the deformations of the plate edges. The double series defines the displacements corresponding to a plate, simply supported along its contour.

The coefficients in (12.5) are again obtained from (12.6), the free terms being given by (12.7). The matrix obtained when only the first nine terms are taken in (12.14) is given in Table 16. The elastic characteristics are determined, as before, by (12.9), and the free terms by (12.11). Tables 13 through 16 permit approximate analysis of a rectangular plate acted upon by an arbitrary external load.

It can be seen that in the general case we have to solve four system of algebraic equations, each containing nine unknowns and having the same structure as the system of canonical equations of the theory of frames. The Gauss method is recommended for this.

After the deflections of the plate have been determined from (12.8), (12.12), (12.13), and (12.14), the bending moments and shearing forces are determined from (1.8) and (1.9). The accuracy of the solution obtained depends on the type of loading and on the number of terms taken in (12.8), (12.12), (12.13), and (12.14). Since trigonometric series converge rapidly in the case of nearly uniformly distributed loads, only nine terms were taken in each series. This approximation is thus satisfactory in practice if the external load is distributed over part of the plate. When greater accuracy is required, the obtained solutions can be extended on the basis of (12.5) and (12.6). If, on the other hand, a lesser accuracy is sufficient, a smaller number of terms can be taken as, for example, in (10.3).

The functions φ_{mn} can also be expressed in different ways, provided the boundary conditions of the problem are satisfied. For example, a high accuracy can be obtained with a small number of terms in (12.3), when the functions φ_{mn} are formed by means of the eigenfunctions of the transverse vibrations of a beam (Table 7) p. 111. Various polynomials may also be chosen as functions φ_{mn} *.

13. GENERAL EQUATIONS FOR THICK PLATES ON ELASTIC SINGLE-LAYER FOUNDATIONS

Consider the three-dimensional deformation of a thick plate on a single-layer foundation (Figure 105).

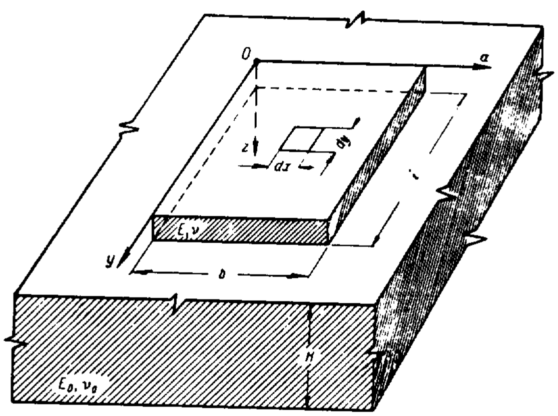


FIGURE 105.

In accordance with the general variational method, the unknown displacements of plate and foundation are assumed to be:

$$\left.\begin{aligned} u(x,\ y,\ z) &= u_1(x,\ y)\,\varphi_1(z), \\ v(x,\ y,\ z) &= v_1(x,\ y)\,\varphi_1(z), \\ w(x,\ y,\ z) &= w_1(x,\ y)\,\psi_1(z) + w_2(x,\ y)\,\psi_2(z), \end{aligned}\right\} \tag{13.1}$$

* Some problems of the analysis of rectangular plates on elastic foundations are discussed by Kosab'yan in /45/.

where

$$u_1(x, y),\ v_1(x, y),\ w_1(x, y),\ w_2(x, y)$$

are unknown functions of x and y,
and

$$\left.\begin{array}{ll} \text{at } z<h & \begin{array}{l} \varphi_1(z) = \dfrac{h-2z}{2}, \\ \psi_1(z) = \dfrac{h-z}{h}, \quad \psi_2(z) = 1; \end{array} \\ \text{at } z>h & \begin{array}{l} \varphi_1(z) = 0, \\ \psi_1(z)=0, \quad \psi_2(z) = \dfrac{H+h-z}{H} \end{array} \end{array}\right\} \qquad (13.2)$$

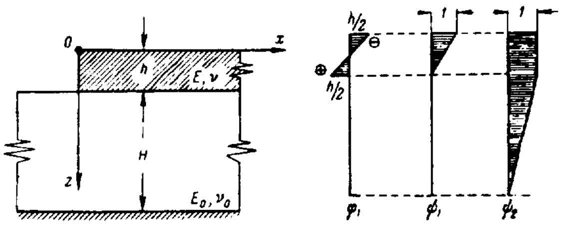


FIGURE 106.

It is seen from (13.2) that $\varphi_1(z)$ and $\psi_1(z)$ define the deformation of a plate on an absolutely rigid foundation. It is assumed that the surface of the foundation is perfectly smooth: no friction or adhesion exists between the plate and the foundation. In contrast to thin plates, vertical compression is taken into account by introducing the function $\psi_1(z)$.

The function $\psi_2(z)$ permits us to make allowance for the elasticity of the foundation: it defines the latter as a single-layer model subjected to both normal stresses σ_z (characteristic of the Winkler foundation) and shearing stresses τ_{zx}, τ_{zy}. For $z>h$, the function $\psi_2(z)$ may be defined in any other way such as a decreasing exponential function or a hyperbolic-sine function (see (11.1)).

The solution given is approximate from the viewpoint of the rigorous mathematical theory of elasticity. The system considered has a finite number of degrees of freedom in the z direction; the horizontal displacements of the elastic foundation are neglected. The solution is, nevertheless, considerably more accurate than that obtained by analyzing a plate on an elastic Winkler foundation, both as regards the strains in the plate itself, as well as those in the elastic foundation.

To determine the unknown functions $u_1(x, y),\ v_1(x, y),\ w_1(x, y), w_2(x, y)$, consider the generalized equilibrium conditions of an elementary column cut from the plate and the elastic foundation (Figure 105). The equilibrium conditions are obtained by equating to zero the work done by all external and internal forces acting on this column over each virtual displacement:

$$\left.\begin{array}{ll} \bar{u}_1(x, y, z) = \varphi_1(z), & \bar{v}_1(x, y, z) = \varphi_1(z), \\ \bar{w}_1(x, y, z) = \psi_1(z), & \bar{w}_2(x, y, z) = \psi_2(z) \\ \text{for} & \\ \quad u_1(x, y) = 1, & v_1(x, y) = 1, \\ \quad w_1(x, y) = 1, & w_2(x, y) = 1. \end{array}\right\} \qquad (13.3)$$

In the general case these equilibrium conditions can be expressed in the form of (6.6) of Chapter I. Substituting (13.1) and (13.3), we obtain:

for $dx = dy = 1$

$$\left.\begin{aligned}
&\int \frac{\partial \sigma_x}{\partial x}\varphi_1\,dz - \int \tau_{xz}\varphi_1'\,dz + \int \frac{\partial \tau_{xy}}{\partial y}\varphi_1\,dz + \int p\varphi_1\,dz = 0,\\
&\int \frac{\partial \sigma_y}{\partial y}\varphi_1\,dz - \int \tau_{yz}\varphi_1'\,dz + \int \frac{\partial \tau_{yx}}{\partial x}\varphi_1\,dz + \int g\varphi_1\,dz = 0,\\
&\int \frac{\partial \tau_{zx}}{\partial x}\varphi_1\,dz - \int \sigma_z\psi_1'\,dz + \int \frac{\partial \tau_{zy}}{\partial y}\psi_1\,dz + \int q\psi_1\,dz = 0,\\
&\int \frac{\partial \tau_{zx}}{\partial x}\psi_2\,dz - \int \sigma_z\psi_2'\,dz + \int \frac{\partial \tau_{zy}}{\partial y}\psi_2\,dz + \int q\psi_2\,dz = 0.
\end{aligned}\right\} \tag{13.4}$$

The integrals in equations (13.4) are taken over the entire height of the elementary column: $0 \leqslant z \leqslant h + H$. The stresses $\sigma_x, \sigma_y, \sigma_z, \tau_{xz}, \tau_{yz}, \tau_{xy}$ are determined by substituting (13.1) in (6.2) of Chapter I, and assuming that the elementary column consists of two layers, whose elastic characteristics are E and ν for $z < h$, and E_0 and ν_0 for $z > h$.

We assume that no body forces act on plate and elastic foundation, and that a vertical surface load $q(x, y)$ is applied to the plate. In this case the load terms in (13.4) are:

$$\left.\begin{aligned}
&\int p\varphi_1\,dz = 0, && \int g\varphi_1\,dz = 0,\\
&\int q\psi_1\,dz = q\psi_1(0) = q, && \int q\psi_2\,dz = q\psi_2(0) = q.
\end{aligned}\right\} \tag{13.5}$$

Substituting (6.2) of Chapter I in (13.4) and taking (13.5) into account, we obtain:

$$\left.\begin{aligned}
&a_{11}\frac{\partial^2 u_1}{\partial x^2} + a_{11}\frac{1-\nu}{2}\frac{\partial^2 u_1}{\partial y^2} - b_{11}\frac{1-\nu}{2}u_1 + a_{11}\frac{1+\nu}{2}\frac{\partial^2 v_1}{dx\,\partial y} -\\
&\qquad - c_{11}\frac{1-\nu}{2}\frac{\partial w_1}{\partial x} - c_{12}\frac{1-\nu}{2}\frac{\partial w_2}{\partial x} = 0,\\
&a_{11}\frac{1+\nu}{2}\frac{\partial^2 u_1}{\partial x\,\partial y} + a_{11}\frac{1-\nu}{2}\frac{\partial^2 v_1}{\partial x^2} + a_{11}\frac{\partial^2 v_1}{\partial y^2} - b_{11}\frac{1-\nu}{2}v_1 -\\
&\qquad - c_{11}\frac{1-\nu}{2}\frac{\partial w_1}{\partial y} - c_{12}\frac{1-\nu}{2}\frac{\partial w_2}{\partial y} = 0,\\
&\left(\frac{1-\nu}{2}c_{11} - \nu d_{11}\right)\frac{\partial u_1}{\partial x} + \left(\frac{1-\nu}{2}c_{11} - \nu d_{11}\right)\frac{\partial v_1}{\partial x} +\\
&\qquad + \frac{1-\nu}{2}r_{11}\frac{\partial^2 w_1}{\partial x^2} + \frac{1-\nu}{2}r_{11}\frac{\partial^2 w_1}{\partial y^2} - s_{11}w_1 +\\
&\qquad + \frac{1-\nu}{2}r_{12}\frac{\partial^2 w_2}{\partial x^2} + \frac{1-\nu}{2}r_{12}\frac{\partial^2 w_2}{\partial y^2} - s_{12}w_2 + \frac{1-\nu^2}{E}q = 0,\\
&\left[\frac{E}{2(1+\nu)}c_{12} - \frac{E}{1-\nu^2}d_{12}\right]\frac{\partial u_1}{\partial x} + \left[\frac{E}{2(1+\nu)}c_{12} - \frac{E}{1-\nu^2}d_{12}\right]\frac{\partial v_1}{\partial x} +\\
&\qquad + \frac{E}{2(1+\nu)}r_{12}\frac{\partial^2 w_1}{dx^2} + \frac{E}{2(1+\nu)}r_{12}\frac{\partial^2 w_1}{\partial y^2} -\\
&\qquad - \frac{E}{1-\nu^2}s_{12}w_1 + \left[\frac{E}{2(1+\nu)}r_{22} + \frac{E_0}{2(1+\nu_0)}r_{22}^0\right]\frac{\partial^2 w_2}{\partial x^2} +\\
&+ \left[\frac{E}{2(1+\nu)}r_{22} + \frac{E_0}{2(1+\nu_0)}r_{22}^0\right]\frac{\partial^2 w_2}{\partial y^2} - \frac{E_0}{1-\nu_0^2}s_{22}^0 w_2 + q = 0,
\end{aligned}\right\} \tag{13.6}$$

where

$$u_1 = u_1(x,\ y),\quad v_1 = v_1(x,\ y),\quad w_1 = w_1(x,\ y),\quad w_2 = w_2(x,\ y);$$

$E,\ \nu,\ E_0,\ \nu_0$ = modulus of elasticity and Poisson's ratio for the plate and the elastic foundation, respectively; $q = q(x,\ y)$ = vertical load acting on the plate.

From (13.2), we obtain for the coefficients in (13.6):

$$\left.\begin{aligned}
a_{11} &= \int \varphi_1^2\,dz = \frac{h^3}{12}, & s_{11} &= \int (\psi_1')^2\,dz = \frac{1}{h},\\
b_{11} &= \int (\varphi_1')^2\,dz = h, & s_{12} &= \int \psi_1'\psi_2'\,dz = 0\\
c_{11} &= \int \varphi_1\psi_1\ dz = -\frac{h}{2}, & s_{22}^0 &= \int (\psi_2')^2\ dz = \frac{1}{H},\\
& & r_{11} &= \int \psi_1^2\ dz = \frac{h}{3},\\
c_{12} &= \int \varphi_1'\psi_2\ dz = -h, & r_{12} &= \int \psi_1\psi_2\ dz = \frac{h}{2},\\
d_{11} &= \int \varphi_1\psi_1'\,dz = 0, & r_{22} &= \int_0^h \psi_2^2\ dz = h,\\
d_{12} &= \int \varphi_1\psi_2'\,dz = 0, & r_{22}^0 &= \int_h^{h+H} \psi_2^2\ dz = \frac{H}{3}.
\end{aligned}\right\} \tag{13.7}$$

Substitution of (13.7) in (13.6) yields the system of differential equations given in Table 17 where the symbols $D_x,\ D_y,\ D_x^2,\ D_y^2$ represent differential operators and indicate that the function written at the top of the column is to be differentiated once or twice by x or y respectively.

This system of four differential equations in the four unknown functions $u_1,\ v_1,\ w_1, w_2$, describes the problem of the bending of a thick plate on an elastic foundation completely. When these functions have been determined, the displacements and stresses in the plate and the elastic foundation can be obtained from (13.1) of this chapter and (6.2) of Chapter I respectively.

TABLE 17

No	u_1	v_1	w_1	w_2	Load term
1	$D_x^2 + \frac{1-\nu}{2} D_y^2 - \frac{6(1-\nu)}{h^2}$	$\frac{1+\nu}{2} D_{xy}^2$	$\frac{3(1-\nu)}{h^2} D_x$	$\frac{6(1-\nu)}{h^2} D_x$	0
2	$\frac{1+\nu}{2} D_{xy}^2$	$\frac{1-\nu}{2} D_x^2 + D_y^2 - \frac{6(1-\nu)}{h^2}$	$\frac{3(1-\nu)}{h^2} D_y$	$\frac{6(1-\nu)}{h^2} D_y$	0
3	$-\frac{1}{2} D_x$	$-\frac{1}{2} D_y$	$\frac{1}{3} D_x^2 + \frac{1}{3} D_y^2 - \frac{2}{(1-\nu)h^2}$	$\frac{1}{2}(D_x^2 + D_y^2)$	$-\frac{2(1+\nu)}{Eh} q$
4	$-D_x$	$-D_y$	$\frac{1}{2}(D_x^2 + D_y^2)$	$\left(1 + \frac{1}{2}\frac{E_0}{E}\frac{1+\nu}{1+\nu_0}\frac{H}{h}\right) D_x^2 + \left(1 + \frac{1}{2}\frac{E_0}{E}\frac{1+\nu}{1+\nu_0}\frac{H}{h}\right) D_y^2 - 2\frac{E_0}{E}\frac{1+\nu}{1+\nu_0}\frac{1}{Hh}$	$-\frac{2(1+\nu)}{Eh} q$

Chapter IV

AXISYMMETRICAL DEFORMATION OF CIRCULAR PLATES ON ELASTIC SINGLE-LAYER FOUNDATIONS

§ 1. STATEMENT OF THE PROBLEM. BASIC DIFFERENTIAL RELATIONSHIPS

1

Consider a circular plate of uniform thickness h resting on an elastic foundation possessing two characteristics (Figure 107). Let the external load be applied symmetrically relative to the plate center, so that the plate is subjected to an axisymmetrical deformation. Polar coordinates (θ, ρ) will be used, the origin of coordinates being placed at the plate center, and the distance from the center to a given point denoted by ρ. The differential equation of bending of a plate resting on an elastic single-layer foundation (cf. (1.5) of Chapter III) is in polar coordinates:

$$\nabla_\rho^2\nabla_\rho^2 W - 2r^2\nabla_\rho^2 W + s^4 W = \frac{p}{D}, \tag{1.1}$$

where

$$\left.\begin{aligned} r^2 &= \frac{E_0}{4(1+\nu_0)D}\int_0^H \psi^2(z)\,dz, \\ s^4 &= \frac{E_0}{(1-\nu_0^2)D}\int_0^H \psi'^2(z)\,dz, \end{aligned}\right\} \tag{1.2}$$

and $D = \frac{Eh^3}{12(1-\mu^2)}$ = flexural rigidity of plate.

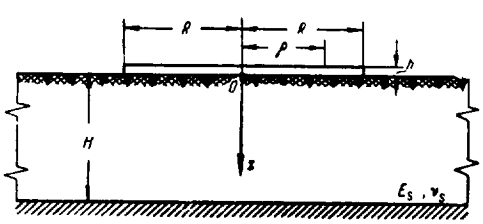


FIGURE 107.

By virtue of the axial symmetry, the plate deflections $W = W(\rho)$ are independent of the polar angle θ, and the Laplacian of W becomes:

$$\nabla_\rho^2 W = \frac{d^2W(\rho)}{d\rho^2} + \frac{1}{\rho}\frac{dW(\rho)}{d\rho}. \tag{1.3}$$

The problem stated is thus completely described by (1.1) and the corresponding boundary conditions. Equation (1.1) differs from the equation of bending of a circular plate on an elastic Winkler foundation by the term:

$$-2r^2\nabla_\rho^2 W$$

through which allowance is made for the work done by the shearing stresses acting in the single-layer foundation.

2

In the case of axisymmetrical bending, radial bending moments M_ρ and shearing forces Q_ρ appear in the cylindrical plate sections $\rho = \mathrm{const}$ (Figure 108). In the radial sections $\theta = \mathrm{const}$ only bending moments M_θ act. The radial and peripheral moments and shearing forces are:

$$\left.\begin{aligned} M_\rho &= -D\left(\frac{d^2W}{d\rho^2} + \frac{\mu}{\rho}\frac{dW}{d\rho}\right) = -D\left[\nabla_\rho^2 W - \frac{1-\mu}{\rho}\frac{dW}{d\rho}\right],\\ M_\theta &= -D\left(\mu\frac{d^2W}{d\rho^2} + \frac{1}{\rho}\frac{dW}{d\rho}\right) = -D\left[\mu\nabla_\rho^2 W + \frac{1-\mu}{\rho}\frac{dW}{d\rho}\right],\\ Q_\rho &= -D\frac{d}{d\rho}\left(\frac{d^2W}{d\rho^2} + \frac{1}{\rho}\frac{dW}{d\rho}\right) = -D\frac{d}{d\rho}\nabla_\rho^2 W. \end{aligned}\right\} \tag{1.4}$$

Shearing forces Q_ρ acting on areas having positive outer normals are considered as positive if their direction coincides with the positive direction of the z axis. Bending moments M_ρ and M_θ causing tension in the lower part of the plate are considered as positive.

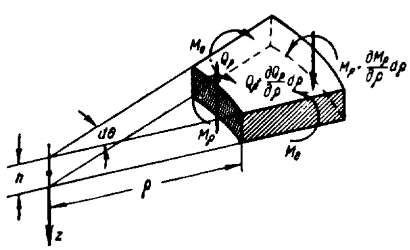


FIGURE 108.

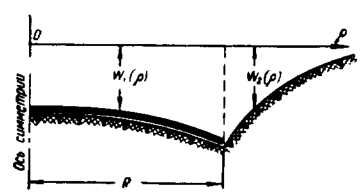


FIGURE 109.

3

We now introduce the generalized shearing force (per unit length) which for $\rho \leqslant R$ defines the shearing stresses in the plate and elastic foundation, acting in the cylindrical sections $\rho = \mathrm{const}$, (cf. (1.10) of Chapter II):

$$N_\rho = -D\left(\frac{d}{d\rho}\nabla_\rho^2 W - 2r^2\frac{dW}{d\rho}\right), \tag{1.5}$$

where

$$r^2 = \frac{t}{D} = \frac{E_0}{4(1+\nu_0)D}\int_0^H \psi^2(z)\,dz.$$

The sign of the forces N_ρ is determined as for the forces Q_ρ.

The generalized shearing force acting in the foundation beyond the plate edge ($\rho \geqslant R$) will be denoted by S_ρ in order to distinguish it from N_ρ. Its value is by (1.5) (cf. (3.10) of Chapter I):

$$S_\rho = 2t\frac{dW}{d\rho}. \tag{1.6}$$

Consider thus a plate with a free edge on which no forces act (Figure 109). The boundary conditions for $\rho = R$ are:

$$M_\rho(R) = 0, \quad W_1(R) = W_2(R), \quad N_\rho(R) = S_\rho(R), \tag{1.7}$$

where W_1 and W_2 are the vertical displacements of plate and surface of elastic foundation for $\rho \leqslant R$ and $\rho > R$ respectively.

The last two conditions (1.7) describe the continuity of the deformed surface of the elastic foundation; through them allowance is made for strains of the elastic foundation beyond the plate edge.

Substitution of (1.5) and (1.6) in the last condition (1.7) yields:

$$-D\frac{d}{d\rho}\nabla_\rho^2 W_1 + 2t\frac{dW_1}{d\rho} = 2t\frac{dW_2}{d\rho}.$$

In virtue of (1.4), this boundary condition can be written:

$$Q_\rho(R) = -D\frac{d}{d\rho}\nabla_\rho^2 W_1 = 2t\left(\frac{dW_2}{d\rho} - \frac{dW_1}{d\rho}\right). \tag{1.8}$$

A fictitious contour force $Q^\Phi = Q_\rho(R)$ thus appears at a free plate edge on which no forces act, due to the coherence of the single-layer foundation and to its capacity for taking up shearing stresses.

We can rewrite (1.8) in the following form*:

$$Q^\Phi = S_{\rho_2} - S_{\rho_1}, \tag{1.9}$$

where S_{ρ_1} and S_{ρ_2} are the generalized shearing forces in the elastic foundation, obtained for sections $\rho = R - \varepsilon$ and $\rho = R + \varepsilon$ respectively, when $\varepsilon \to 0$.

§2. GENERAL INTEGRAL OF THE DIFFERENTIAL EQUATION FOR A CIRCULAR PLATE ON A SINGLE-LAYER FOUNDATION

1

We replace ρ by the dimensionless coordinate $\xi = \frac{\rho}{L_0}$, where:

$$L_0 = \sqrt[4]{\frac{D}{k}}, \tag{2.1}$$

* The fictitious forces Q^Φ were similarly defined in the analysis of beams (see section 5, Chapter II).

and $k = \frac{E_0}{1-\nu_0^2}\int_0^H \psi'^2(z)\,dz$ defines the compressive strain of the soil, D = = flexural rigidity of plate.

We can now write (1.1) in the form:

$$\nabla_\xi^2\nabla_\xi^2 W - 2r_0^2\nabla_\xi^2 W + W = \frac{pL_0^4}{D}, \qquad (2.2)$$

where

$$r_0^2 = \frac{tL_0^2}{D} = \frac{E_0 L_0}{4(1+\nu_0)D}\int_0^H \psi^2(z)\,dz, \qquad (2.3)$$

and

$$\nabla_\xi^2 = \frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi}. \qquad (2.4)$$

2

When no surface load acts, (2.2) reduces to the homogeneous equation:

$$\nabla_\xi^2\nabla_\xi^2 W - 2r_0^2\nabla_\xi^2 W + W = 0. \qquad (2.5)$$

This can be reduced to an equivalent system of two second-order differential equations. Let $W = W(\xi)$ be a particular solution of (2.5) satisfying at the same time the differential equation

$$\nabla_\xi^2 W + \lambda W = 0, \qquad (2.6)$$

where λ is a constant to be determined.

It follows from (2.6) that

$$\left.\begin{aligned} \nabla_\xi^2 W &= -\lambda W, \\ \nabla_\xi^2\nabla_\xi^2 W &= \lambda^2 W. \end{aligned}\right\} \qquad (2.7)$$

Substitution of these expressions in (2.5) yields the following equation for λ:

$$\lambda^2 + 2r_0^2\lambda + 1 = 0. \qquad (2.8)$$

Its roots are

$$\left.\begin{aligned} \lambda_1 &= -r_0^2 + \sqrt{(r_0^2)^2 - 1}, \\ \lambda_2 &= -r_0^2 - \sqrt{(r_0^2)^2 - 1}. \end{aligned}\right\} \qquad (2.9)$$

For actual soils

$$0 \leqslant r_0^2 < 1. \qquad (2.10)$$

The case $r_0^2 = 0$ is limiting; it characterizes the absence of shearing forces in the elastic foundation $(t = 0)$.

The roots of (2.8) are therefore conjugate complex numbers:

$$\left.\begin{aligned}\lambda_1 &= a = -a_1 + b_1 i,\\ \lambda_2 &= \bar{a} = -a_1 - b_1 i,\end{aligned}\right\} \tag{2.11}$$

where

$$a_1 = r_0^2, \quad b_1 = \sqrt{1 - (r_0^2)^2}.$$

The complex numbers (2.11) may have the following values (Figure 110):

$$\left.\begin{aligned}\pi > \arg a \geqslant \frac{\pi}{2},\\ -\pi < \arg \bar{a} \leqslant -\frac{\pi}{2}\end{aligned}\right\} \tag{2.12}$$

In accordance with Viéte's theorem, their modulus is equal to the free term in (2.8), i.e., to unity:

$$|a| = 1 \quad |\bar{a}| = 1. \tag{2.13}$$

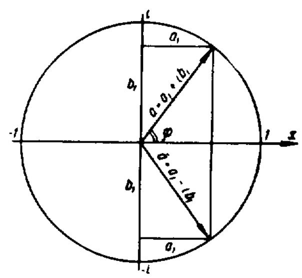


FIGURE 110.

Proceeding from (2.6), we find that the following two independent second-order differential equations correspond to the conjugate complex roots (2.11):

$$\left.\begin{aligned}\frac{d^2W_1}{d\xi^2} + \frac{1}{\xi}\frac{dW_1}{d\xi} + aW_1 = 0,\\ \frac{d^2W_2}{d\xi^2} + \frac{1}{\xi}\frac{dW_2}{d\xi} + \bar{a}W_2 = 0.\end{aligned}\right\} \tag{2.14}$$

3

The general integral of (2.2) can now be written in the form:

$$W = W_1 + W_2 + W_p, \tag{2.15}$$

where W_1 and W_2 satisfy the first and the second equations (2.14) respectively and W_p is a particular integral of the nonhomogeneous equation (2.2).

By introducing the new variables:

$$u = \sqrt{a}\,\xi, \quad v = \sqrt{\bar{a}}\,\xi, \tag{2.16}$$

we can transform (2.14) into zero-order Bessel equations:

$$\left.\begin{aligned} \frac{d^2W_1}{du^2} + \frac{1}{u}\frac{dW_1}{du} + W_1 = 0, \\ \frac{d^2W_2}{dv^2} + \frac{1}{v}\frac{dW_2}{dv} + W_2 = 0. \end{aligned}\right\} \tag{2.17}$$

The solution of (2.17) can be represented in the following form*:

$$\left.\begin{aligned} W_1(\xi) = B_1J_0(\sqrt{a}\,\xi) + B_2H_0^{(1)}(\sqrt{a}\,\xi), \\ W_2(\xi) = B_3J_0(\sqrt{\bar{a}}\,\xi) + B_4H_0^{(2)}(\sqrt{\bar{a}}\,\xi), \end{aligned}\right\} \tag{2.18}$$

where

$$J_0(\sqrt{a}\,\xi) \text{ and } J_0(\sqrt{\bar{a}}\,\xi)$$

are zero-order Bessel functions of the first kind in $\sqrt{a}\,\xi$ and $\sqrt{\bar{a}}\,\xi$; and:

$$H_0^{(1)}(\sqrt{a}\,\xi) \text{ and } H_0^{(2)}(\sqrt{\bar{a}}\,\xi)$$

are zero-order Hankel functions of the first and second kind respectively, also in $\sqrt{a}\,\xi$ and $\sqrt{\bar{a}}\,\xi$.

Using (2.18), we can write (2.15) in the following final form:

$$\begin{aligned} W = B_1J_0(\sqrt{a}\,\xi) + B_2H_0^{(1)}(\sqrt{a}\,\xi) + \\ + B_3J_0(\sqrt{\bar{a}}\,\xi) + B_4H_0^{(2)}(\sqrt{\bar{a}}\,\xi) + W_p. \end{aligned} \tag{2.19}$$

For the solution of practical problems it is convenient to write:

$$\left.\begin{aligned} \sqrt{a} = e^{i\varphi} = \cos\varphi + i\sin\varphi, \\ \sqrt{\bar{a}} = e^{-i\varphi} = \cos\varphi - i\sin\varphi, \end{aligned}\right\} \tag{2.20}$$

where

$$\varphi = \frac{1}{2}\arg a$$

and the modulus of the complex numbers $\sqrt{a}$ and $\sqrt{\bar{a}}$ is equal to unity in accordance with (2.13).

It can be seen from (2.12) and (2.20) that the functions:

$$J_0(\sqrt{a}\,\xi), \quad H_0^{(1)}(\sqrt{a}\,\xi), \quad J_0(\sqrt{\bar{a}}\,\xi), \quad H_0^{2}(\sqrt{\bar{a}}\,\xi)$$

are determined in the regions:

$$\frac{\pi}{2} > \varphi \geqslant \frac{\pi}{4}, \qquad -\frac{\pi}{2} < \varphi \leqslant -\frac{\pi}{4}. \tag{2.21}$$

In the particular case $t = 0$, $(r_0^2 = 0)$, these functions are determined along a line forming an angle $\varphi = \frac{\pi}{4}$ with the axis of real magnitudes.

* For a thorough treatment of the theory of Bessel functions, see /4/.

Since the functions

$$J_0(\sqrt{a}\xi),\ H_0^{(1)}(\sqrt{a}\xi),\ J_0(\sqrt{\bar{a}}\xi),\ H_0^{(2)}(\sqrt{\bar{a}}\xi)$$

are complex while the plate-deflection function W must be real, the constants C_1, C_2, C_3, C_4 must be complex. In order to express the solution through real functions, we write (2.19) in a different form:

$$W = C_1 u_0(\xi) + C_2 v_0(\xi) + C_3 f_0(\xi) + C_4 g_0(\xi) + W_p, \tag{2.22}$$

where, as before, W_p is a particular integral of the nonhomogeneous equation (2.2), and

$$\left.\begin{aligned} u_0(\xi) &= \operatorname{Re} J_0(\sqrt{a}\xi) = \frac{J_0(\sqrt{a}\xi) + J_0(\sqrt{\bar{a}}\xi)}{2}, \\ v_0(\xi) &= \operatorname{Im} J_0(\sqrt{a}\xi) = \frac{J_0(\sqrt{a}\xi) - J_0(\sqrt{\bar{a}}\xi)}{2i}, \\ f_0(\xi) &= \operatorname{Re} H_0^{(1)}(\sqrt{a}\xi) = \frac{H_0^{(1)}(\sqrt{a}\xi) + H_0^{(2)}(\sqrt{\bar{a}}\xi)}{2}, \\ g_0(\xi) &= \operatorname{Im} H_0^{(2)}(\sqrt{\bar{a}}\xi) = \frac{H_0^{(1)}(\sqrt{a}\xi) - H_0^{(2)}(\sqrt{\bar{a}}\xi}{2i}. \end{aligned}\right\} \tag{2.23}$$

It is seen from (2.23) that $u_0(\xi)$ and $f_0(\xi)$ represent the real, $v_0(\xi)$ and $g_0(\xi)$, the imaginary parts of the zero-order Bessel and Hankel functions. Since these functions are real, the constants C_1, C_2, C_3, C_4 will also be real. The behavior of functions $u_0(\xi)$ and $v_0(\xi)$ resembles that of the functions $e^{\xi}\cos\xi$, $e^{\xi}\sin\xi$ appearing in the theory of beams on elastic Winkler foundations: they remain finite when $\xi \to 0$, and tend to infinity when $\xi \to \infty$.

At $\xi \to 0$, the function $f_0(\xi)$ has a singularity of the type $\xi^2 \ln \xi$; the function $g_0(\xi)$ becomes infinite when $\xi \to 0$. Both functions tend to zero when $\xi \to \infty$, resembling the functions $e^{-\xi}\cos\xi$, $e^{-\xi}\sin\xi$.

The following expressions are obtained for the slopes, bending moments, and shearing forces in the plate by substituting (2.22) into (1.4) and using the known rules of differentiation of cylindrical functions:

$$\left.\begin{aligned} \frac{dW}{d\rho} &= -\frac{1}{L_0}\left[C_1\theta_1(\xi) + C_2\theta_2(\xi) + C_3\theta_3(\xi) + C_4\theta_4(\xi) - \frac{dW_p}{d\xi}\right], \\ M_\rho &= \frac{D}{L_0^2}\Big\{C_1[M_1(\xi) - (1-\mu)\overline{M}_1(\xi)] + C_2[M_2(\xi) - \\ &\quad - (1-\mu)\overline{M}_2(\xi)] + C_3[M_3(\xi) - (1-\mu)\overline{M}_3(\xi)] + \\ &\quad + C_4[M_4(\xi) - (1-\mu)\overline{M}_4(\xi)] - \left[\nabla_\xi^2 - \frac{1-\mu}{\xi}\frac{d}{d\xi}\right]W_p\Big\}, \\ M_0 &= \frac{D}{L_0^2}\Big\{C_1[\mu M_1(\xi) + (1-\mu)\overline{M}_1(\xi)] + C_2[\mu M_2(\xi) + \\ &\quad + (1-\mu)\overline{M}_2(\xi)] + C_3[\mu M_3(\xi) + (1-\mu)\overline{M}_3(\xi)] + \\ &\quad + C_4[\mu M_4(\xi) + (1-\mu)\overline{M}_4(\xi)] - \left[\mu\nabla_\xi^2 + \frac{1-\mu}{\xi}\frac{d}{d\xi}\right]W_p\Big\}, \\ Q_\rho &= -\frac{D}{L_0^3}\left[C_1Q_1(\xi) + C_2Q_2(\xi) + C_3Q_3(\xi) + C_4Q_4(\xi) + \frac{d}{d\xi}\nabla_\xi^2 W_p\right]. \end{aligned}\right\} \tag{2.24}$$

where

$$\left.\begin{aligned}\theta_1(\xi) &= u_1(\xi)\cos\varphi - v_1(\xi)\sin\varphi,\\ \theta_2(\xi) &= u_1(\xi)\sin\varphi + v_1(\xi)\cos\varphi,\\ \theta_3(\xi) &= f_1(\xi)\cos\varphi - g_1(\xi)\sin\varphi,\\ \theta_4(\xi) &= f_1(\xi)\sin\varphi + g_1(\xi)\cos\varphi;\end{aligned}\right\} \tag{2.25}$$

$$\left.\begin{aligned}M_1(\xi) &= u_0(\xi)\cos 2\varphi - v_0(\xi)\sin 2\varphi,\\ M_2(\xi) &= u_0(\xi)\sin 2\varphi + v_0(\xi)\cos 2\varphi,\\ M_3(\xi) &= f_0(\xi)\cos 2\varphi - g_0(\xi)\sin 2\varphi,\\ M_4(\xi) &= f_0(\xi)\sin 2\varphi + g_0(\xi)\cos 2\varphi;\end{aligned}\right\} \tag{2.26}$$

$$\left.\begin{aligned}\bar{M}_1(\xi) &= \frac{1}{\xi}\,[u_1(\xi)\cos\varphi - v_1(\xi)\sin\varphi],\\ \bar{M}_2(\xi) &= \frac{1}{\xi}\,[u_1(\xi)\sin\varphi + v_1(\xi)\cos\varphi],\\ \bar{M}_3(\xi) &= \frac{1}{\xi}\,[f_1(\xi)\cos\varphi - g_1(\xi)\sin\varphi],\\ \bar{M}_4(\xi) &= \frac{1}{\xi}\,[f_1(\xi)\sin\varphi + g_1(\xi)\cos\varphi];\end{aligned}\right\} \tag{2.27}$$

$$\left.\begin{aligned}Q_1(\xi) &= u_1(\xi)\cos 3\varphi - v_1(\xi)\sin 3\varphi,\\ Q_2(\xi) &= u_1(\xi)\sin 3\varphi + v_1(\xi)\cos 3\varphi,\\ Q_3(\xi) &= f_1(\xi)\cos 3\varphi - g_1(\xi)\sin 3\varphi,\\ Q_4(\xi) &= f_1(\xi)\sin 3\varphi + g_1(\xi)\cos 3\varphi,\end{aligned}\right\} \tag{2.28}$$

and

$$\varphi = \frac{1}{2}\arg a; \tag{2.29}$$

the complex number a is given by (2.11).

The functions $u_1(\xi)$, $v_1(\xi)$, and $f_1(\xi)$, $g_1(\xi)$ represent the real and the imaginary parts of the first-order Bessel and Hankel functions respectively, and are determined from the functions

$$J_1(\sqrt{a}\xi),\ J_1(\sqrt{\bar{a}}\xi),\ H_1^{(1)}(\sqrt{a}\xi),\ H_1^{(2)}(\sqrt{\bar{a}}\xi)$$

in a manner similar to (2.23).

6

Expressions (2.22) and (2.24) are the general solution to the problem of the axisymmetrical deformation of a circular plate on a single-layer foundation. The integration constants C_1, C_2, C_3, C_4 must be determined from the boundary conditions. Establishing the latter presents no difficulty in the usual cases of plate support (simple support or built-in edge). Thus, for a simple support along the edge,

$$\left.\begin{aligned}W &= \left(\frac{R}{L_0}\right) = 0,\\ M_\rho\left(\frac{R}{L_0}\right) &= 0;\end{aligned}\right\} \tag{2.30}$$

for a built-in edge

$$\left.\begin{aligned} W = \left(\frac{R}{L_0}\right) = 0, \\ \left(\frac{dW}{d\rho}\right)_{\text{при } \xi = \frac{R}{L_0}} = 0. \end{aligned}\right\} \qquad (2.31)$$

If the plate is freely supported on the elastic foundation, the coherence of the soil and the possibility of strains appearing in it beyond the region of load application necessitate a consideration of an infinite region lying beyond the plate edge. As was shown in section 1 of this chapter, this is expressed through a fictitious shearing force Q^{Φ} acting along the plate edge, which has to be taken into account in the boundary conditions.

§ 3. ABSOLUTELY RIGID PLATE

Consider a circular plate under the action of an axisymmetrical load whose resultant is P_0. Let the plate be so rigid that its deformations can be neglected; it can then be considered as a circular punch whose displacement is $W_1 = C_0$ (Figure 111).

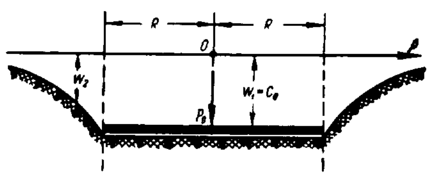


FIGURE 111.

The states of strain and stress of the elastic foundation beyond the plate edges ($R \leqslant \rho < \infty$) are determined in the general case by (7.8) of Chapter I. When no surface loads act within the region considered, this equation reduces to the homogeneous equation:

$$\frac{d^2W_2}{d\rho^2} + \frac{1}{\rho}\frac{dW_2}{d\rho} - \alpha^2 W_2 = 0, \qquad (3.1)$$

where $\alpha = \sqrt{\frac{k}{2t}}$, and $W_2 = W_2(\rho)$ is a function characterizing the vertical displacements of the foundation beyond the plate edges.

The solution of (3.1) (cf. section 7 of Chapter I) is:

$$W_2 = C_1 I_0(\alpha\rho) + C_2 K_0(\alpha\rho). \qquad (3.2)$$

The problem is thus reduced to determining the integration constants C_1 and C_2, as well as the vertical displacement of the plate C_0, from the boundary and equilibrium conditions of the plate.

Since the deformed surface of the elastic foundation is assumed to be continuous, while the vertical displacements of the foundation are equal to zero at infinity, the boundary conditions are:

$$\left.\begin{array}{ll} \text{at } \rho = R: & W_2 = C_0; \\ \text{at } \rho \to \infty: & W_2 \to 0. \end{array}\right\} \tag{3.3}$$

Taking into account the behavior of the function $I_0(\alpha\rho)$ at infinity, the second condition yields:

$$C_1 = 0. \tag{3.4}$$

The first condition gives then:

$$C_2 = \frac{C_0}{K_0(\alpha R)}. \tag{3.5}$$

To determine C_0, the equilibrium condition of the system (plate + elastic foundation) will now be formulated by equating to zero the total work done by all external and internal forces acting on the system over the virtual displacement:

$$\bar{w}(\rho, z) = 1 \cdot \psi(z),$$

We obtain:

$$-\left[\int_0^H\int_0^{2\pi}\int_0^R \sigma_{z_1}\psi'(z)\,\rho\,d\rho\,d\theta\,dz + \int_0^H\int_0^{2\pi}\int_R^\infty \sigma_{z_2}\psi'(z)\,\rho\,d\rho\,d\theta\,dz\right] + \\ + P_0\psi(0) = 0, \tag{3.6}$$

where σ_{z_1} and σ_{z_2} are the normal stresses appearing in the elastic foundation beneath the plate and beyond its edges respectively. According to (6.4) of Chapter I, these stresses are:

$$\left.\begin{array}{l} \sigma_{z_1} = \dfrac{E_0}{1-\nu_0^2} C_0\psi'(z), \\ \sigma_{z_2} = \dfrac{E_0}{1-\nu_0^2} W_2(\rho)\,\psi'(z). \end{array}\right\} \tag{3.7}$$

Substitution of these expressions in (3.6) yields, after integrating between the limits shown:

$$kC_0\left[\pi R^2 + 2\pi R^2 \frac{K_1(\alpha R)}{\alpha R K_0(\alpha R)}\right] = P_0,$$

or finally:

$$C_0 = \frac{P_0}{\pi R^2 k\left[1 + 2\dfrac{K_1(\alpha R)}{\alpha R K_0(\alpha R)}\right]}, \tag{3.8}$$

where, as before:

$$k = \frac{E_0}{1-\nu_0^2}\int_0^H \psi'^2\,dz. \tag{3.9}$$

The reactions of the elastic foundation can be obtained from (7.8) of Chapter I by putting $W_1 = C_0$:

$$q = kC_0 = \frac{P_0}{\pi R^2\left[1 + 2\frac{K_1(\alpha R)}{\alpha R K_0(\alpha R)}\right]}. \tag{3.10}$$

In addition to the distributed reactions (3.10), fictitious reactions Q^Φ, whose dimensions are kg/cm or t/m, act along the contour of the circular plate (Figure 112). These are due to the strains of the elastic foundation beyond the plate edges and correspond to the infinitely large pressures beneath the edges of the circular punch, found by the exact methods of the theory of elasticity.

The fictitious reactions Q^Φ can be determined from (1.8), putting $W_1 = C_0$:

$$Q^\Phi = 2t\left(\frac{dW_2}{d\rho}\right)_{\rho=R}. \tag{3.11}$$

Substitution of (3.2), (3.4), (3.5), and (3.8) leads to the following final expression for Q^Φ:

$$Q^\Phi = \frac{P_0}{\pi R\left[1 + 2\frac{K_1(\alpha R)}{\alpha R K_0(\alpha R)}\right]}\,\frac{K_1(\alpha R)}{\alpha R K_0(\alpha R)}. \tag{3.12}$$

The reactions of the elastic foundation, obtained from (3.10) and (3.12), satisfy the static-equilibrium condition of the plate $\sum Z = 0$. It is easily seen that:

$$\pi R^2 q + 2\pi R Q^\Phi = P_0. \tag{3.13}$$

This solution is true for any function $\psi(z)$, with the same accuracy as that with which:

$$\alpha = \sqrt{\frac{k}{2t}} \tag{3.14}$$

has been obtained. In many practical problems it is convenient to select $\psi(z)$ in the linear form (2.7) of Chapter I, or in the form:

$$\psi(z) = \frac{\operatorname{sh}\frac{\gamma(H-z)}{R}}{\operatorname{sh}\frac{\gamma H}{R}}, \tag{3.15}$$

where H = thickness of compressible soil layer, R = plate radius, and γ = dimensionless coefficient depending on the elastic properties of the foundation. When $\psi(z)$ is given by (3.15), the integral characteristics of the elastic foundation are [cf. (7.11) of Chapter I]:

$$\left.\begin{aligned} k &= \frac{E_0}{(1-\nu_0^2)H}\psi_k, \\ t &= \frac{E_0 H}{12(1+\nu_0)}\psi_t, \\ \alpha &= \frac{1}{H}\sqrt{\frac{6}{1-\nu_0}}\psi_\alpha, \end{aligned}\right\} \tag{3.16}$$

where [cf. Chapter I, (7.12)]:

$$\left.\begin{aligned}
\psi_k &= \frac{1}{2}\frac{\gamma H}{R}\left[\frac{\operatorname{sh}\frac{\gamma H}{R}\operatorname{ch}\frac{\gamma H}{R}+\frac{\gamma H}{R}}{\operatorname{sh}^2\frac{\gamma H}{R}}\right],\\
\psi_t &= \frac{3}{2}\frac{R}{\gamma H}\left[\frac{\operatorname{sh}\frac{\gamma H}{R}\operatorname{ch}\frac{\gamma H}{R}-\frac{\gamma H}{R}}{\operatorname{sh}^2\frac{\gamma H}{R}}\right],\\
\varphi_a &= \frac{\gamma H}{R}\sqrt{\frac{1}{3}\,\frac{\operatorname{sh}\frac{\gamma H}{R}\operatorname{ch}\frac{\gamma H}{R}+\frac{\gamma H}{R}}{\operatorname{sh}\frac{\gamma H}{R}\operatorname{ch}\frac{\gamma H}{R}-\frac{\gamma H}{R}}}.
\end{aligned}\right\} \tag{3.17}$$

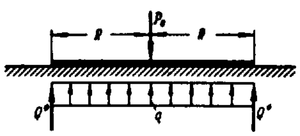


FIGURE 112.

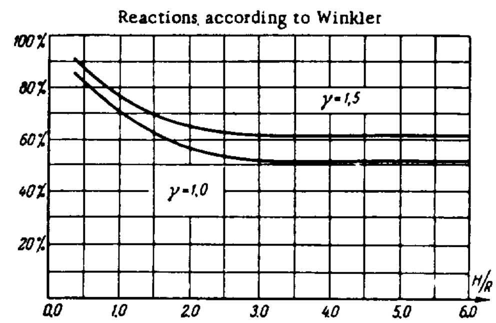


FIGURE 113.

Curves of q as a function of the reduced thickness $\frac{H}{R}$ are easily plotted, using (3.10), (3.16), and (3.17), as in Figure 113 for $\nu_0 = 0.4$ and $\gamma = 1.0$, $\gamma = 1.5$. In this diagram the ordinate defines the ratio (in %) between q and the corresponding values of the reactive pressure as given by Winkler. It is seen that for a single-layer foundation acting like the base of a press, q is less than the value according to Winkler. This is due to the fictitious forces Q^Φ acting along the contour, which characterize the state of strain of the single-layer foundation beyond the plate edges; they are equivalent to the infinite stresses obtained in the exact solution of the theory of elasticity.

These curves also show that q becomes practically constant for $\frac{H}{R} > 2.5$, so that, for $\frac{H}{R} > 2.5$, the foundation can be considered as a semi-infinite elastic space ($H = \infty$). For $\frac{H}{R} < 1.0$, on the other hand, the single-layer foundation approximates Winkler's model in its behavior, the distributed reactions q increase while the concentrated reactions Q^Φ decrease.

After the reactions of the elastic foundation have been found, the bending moments and shearing forces acting on the rigid plate can be determined by the ordinary methods applied to symmetrically loaded circular plates. In this case the external load consists of the given actual load and the reactions of the elastic foundation.

The following expressions are thus obtained for the radial bending moments at the plate center, induced by the distributed reactions q and the concentrated reaction Q^{Φ}:

$$M_q(0) = qR^2\overline{M}_q(0), \quad M_Q(0) = PR\overline{M}_Q(0), \tag{3.18}$$

where, using the same notations as in (3.10) and (3.12):

$$\left.\begin{aligned} \overline{M}_q(0) &= \frac{(3+\mu)\dfrac{K_1(\alpha R)}{\alpha R K_0(\alpha R)}}{8\left[1+2\dfrac{K_1(\alpha R)}{\alpha R K_0(\alpha R)}\right]}, \\ \overline{M}_Q(0) &= -\frac{3+\mu}{8\left[1+2\dfrac{K_1(\alpha R)}{\alpha R K_0(\alpha R)}\right]} \end{aligned}\right\} \tag{3.19}$$

Curves of $\overline{M}_q(0)$ and $\overline{M}_Q(0)$ as functions of $\frac{H}{R}$, obtained from these formulas, have been plotted in Figure 114 for the function $\psi(z)$ given by (3.15). The elastic characteristics of the foundation are defined by (3.16) and (3.17). The following numerical values were used: $\nu_0 = 0.4$, $\mu = 1/6$, $\gamma = 1.0$ and $\gamma = 1.5$.

It is seen that the radial bending moments at the plate center increase with the depth of the elastic layer, tending toward a finite value; for $\gamma = 1.5$ and $\frac{H}{R} > 2.0$ the solution presented is practically identical with the solution given by the theory of the semi-infinite elastic space; for $\gamma = 1.0$ the difference is about 20 to 25%.

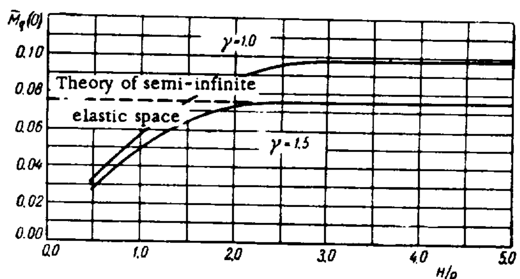


FIGURE 114.

§ 4. ANNULAR PUNCH

Consider an annular punch subjected to an axisymmetrical load P, distributed along the circumference of the circle $\rho = R$ (Figure 115), whose resultant is:

$$P_0 = 2\pi RP.$$

The inner and outer radii of the annulus will be denoted by R_1 and R_2 respectively. The following notations will be used for the vertical displacements of the elastic-foundation surface: $W_1(\rho)$ = vertical displacement inside annulus, $W_2 = C_0$ = vertical displacement beneath annulus, $W_3(\rho)$ = vertical displacement outside annulus.

The homogeneous differential equation (3.2) holds true for $0 \leqslant \rho \leqslant R_1$ and $R_2 \leqslant \rho < \infty$; the vertical displacements in these regions are thus:

$$\left.\begin{aligned} W_1(\rho) &= C_1 I_0(\alpha\rho) + C_2 K_0(\alpha\rho), \\ W_3(\rho) &= C_3 I_0(\alpha\rho) + C_4 K_0(\alpha\rho). \end{aligned}\right\} \tag{4.1}$$

The boundary conditions are:

$$\left.\begin{aligned} &\text{at } \rho = 0: && \frac{dW_1}{d\rho} = 0; \\ &\text{at } \rho = R_1: && W_1 = C_0; \\ &\text{at } \rho = R_2: && W_3 = C_0; \\ &\text{at } \rho \to \infty: && W_3 = 0. \end{aligned}\right\} \tag{4.2}$$

The constants C_1, C_2, C_3, C_4 are, by (4.1) and (4.2):

$$\left.\begin{aligned} &C_1 = \frac{C_0}{I_0(\alpha R_1)}, \quad C_4 = \frac{C_0}{K_0(\alpha R_2)}. \\ &C_2 = C_3 = 0, \end{aligned}\right\} \tag{4.3}$$

The vertical displacements of the surface of the elastic foundation are, therefore:

$$\left.\begin{aligned} W_1(\rho) &= \frac{C_0}{I_0(\alpha R_1)} I_0(\alpha\rho), \\ W_2(\rho) &= C_0, \\ W_3(\rho) &= \frac{C_0}{K_0(\alpha R_2)} K_0(\alpha\rho). \end{aligned}\right\} \tag{4.4}$$

We determine C_0 by the equilibrium conditions of the system considered, applying Lagrange's principle of virtual displacements. We obtain by analogy with (3.6):

$$\begin{aligned} -\Bigg[\int_0^H \int_0^{2\pi} \int_0^{R_1} \sigma_{z_1}\psi'(z)\,\rho\,d\rho\,d\theta\,dz + \int_0^H \int_0^{2\pi} \int_{R_1}^{R_2} \sigma_{z_2}\psi'(z)\,\rho\,d\rho\,d\theta\,dz + \\ + \int_0^H \int_0^{2\pi} \int_{R_2}^{\infty} \sigma_{z_3}\psi'(z)\,\rho\,d\rho\,d\theta\,dz\Bigg] + P_0\psi_1(0) = 0. \end{aligned} \tag{4.5}$$

The expression in brackets appears with the minus sign since it represents the work done by the internal forces.

Substituting the values of the normal stresses, given by expressions similar to (3.7), and integrating, we obtain:

$$C_0 = \frac{P_0}{k\left[\pi(R_2^2 - R_1^2) + 2\pi R_1^2 \frac{I_1(\alpha R_1)}{I_0(\alpha R_1)\,\alpha R_1} + 2\pi R_2^2 \frac{K_1(\alpha R_2)}{K_0(\alpha R_2)\,\alpha R_2}\right]}. \tag{4.6}$$

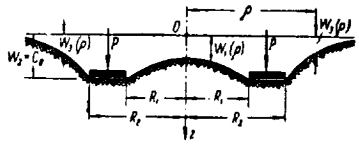


FIGURE 115.

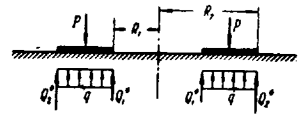


FIGURE 116.

Using (4.4) and (4.6), the reactions of the elastic foundation are, by (7.8) of Chapter I and (1.9) of this chapter:

$$q = \frac{P_0}{\pi (R_2^2 - R_1^2)\left[1 + 2\left(\frac{R_1^2}{R_2^2 - R_1^2}\frac{I_1(\alpha R_1)}{\alpha R_1 I_0(\alpha R_1)} + \frac{R_2^2}{R_2^2 - R_1^2}\frac{K_1(\alpha R_2)}{\alpha R_2 K_0(\alpha R_2)}\right)\right]},$$

$$Q_1^{\Phi} = \frac{P_0 R_1 \frac{I_1(\alpha R_1)}{\alpha R_1 I_0(\alpha R_1)}}{\pi (R_2^2 - R_1^2)\left[1 + 2\left(\frac{R_1^2}{R_2^2 - R_1^2}\frac{I_1(\alpha R_1)}{\alpha R_1 I_0(\alpha R_1)} + \frac{R_2^2}{R_2^2 - R_1^2}\frac{K_1(\alpha R_2)}{\alpha R_2 K_0(\alpha R_2)}\right)\right]},$$

$$Q_2^{\Phi} = \frac{P_0 R_2 \frac{K_1(\alpha R_2)}{\alpha R_2 K_0(\alpha R_2)}}{\pi (R_2^2 - R_1^2)\left[1 + 2\left(\frac{R_1^2}{R_2^2 - R_1^2}\frac{I_1(\alpha R_1)}{\alpha R_1 I_0(\alpha R_1)} + \frac{R_2^2}{R_2^2 - R_1^2}\frac{K_1(\alpha R_2)}{\alpha R_2 K_0(\alpha R_2)}\right)\right]}.$$

Here, Q_1^{Φ} and Q_2^{Φ} are fictitious forces acting along the inner and outer punch contours respectively (Figure 116).

5. INFINITE PLATE UNDER THE ACTION OF A CONCENTRATED FORCE

Consider an infinite plate loaded by a concentrated force P (Figure 117). The origin of coordinates is located at the point of application of the force. The problem is then one of axisymmetrical loading, and can be described by the homogeneous differential equation:

$$D\nabla_\rho^2\nabla_\rho^2 W - 2t\nabla_\rho^2 W + kW = 0. \qquad (5.1)$$

As was shown in section 2, the solution of (5.1) can be represented in the form:

$$W = C_1 u_0(\xi) + C_2 v_0(\xi) + C_3 f_0(\xi) + C_4 g_0(\xi), \quad \text{[cf. (2.22)]} \qquad (5.2)$$

where

$$\xi = \frac{\rho}{L_0}, \text{ and } L_0 = \sqrt[4]{\frac{D}{k}}.$$

The solution thus reduces to determining the integration constants C_1, C_2, C_3, C_4.

All forces and displacements must tend to zero when $\xi \to \infty$. Since the functions $u_0(\xi)$ and $v_0(\xi)$ tend to infinity when $\xi \to \infty$, then

$$C_1 = C_2 = 0. \tag{5.3}$$

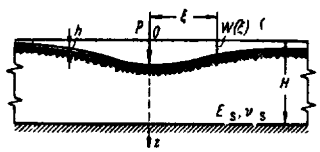


FIGURE 117.

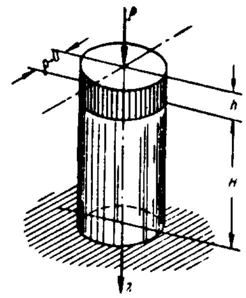


FIGURE 118.

The plate deflection must remain finite at the origin ($\xi = 0$). Since the function $f_0(\xi)$ remains finite when $\xi \to 0$, while $g_0(\xi)$ tends to infinity, the coefficient of $g_0(\xi)$ must be zero: $C_4 = 0$.

It therefore follows that:

$$W = C_3 f_0(\xi). \tag{5.4}$$

To determine the constant C_3, consider the equilibrium condition of an infinitesimal cylinder ($\rho \to 0$) cut out of the plate and the elastic foundation at the origin of coordinates (Figure 118). Applying the variational principle, we obtain:

$$\int_0^{2\pi} N_\rho \rho \, d\theta + P = 0, \tag{5.5}$$

where N_ρ = generalized shearing force, given by (1.5).

The generalized equilibrium condition (5.5) could also have been written in the form:

$$\int_0^{2\pi} Q_\rho \rho d\theta + P = 0, \tag{5.6}$$

since, for reasons of symmetry, we have:

at $\rho = 0$
$$\frac{dW}{d\rho} = 0. \tag{5.7}$$

The last equation (2.24) yields for the shearing force:

$$Q_\rho = -\frac{D}{L_0^3} C_3 Q_3(\xi). \tag{5.8}$$

Substituting (5.8) in (5.6) and integrating, we obtain:

$$C_3 = \frac{PL_0^2}{4D \sin 2\varphi}. \tag{5.9}$$

From (5.4) and (2.24), we finally obtain the displacements, moments, and forces for an infinite plate:

$$\left.\begin{aligned}
W &= \frac{PL_0^2}{4D\sin 2\varphi} f_0(\xi),\\
\frac{dW}{d\rho} &= -\frac{PL_0}{4D\sin 2\varphi}\theta_3(\xi),\\
M_\rho &= \frac{P}{4\sin 2\varphi}\left[M_3(\xi) - (1-\mu)\overline{M}_3(\xi)\right],\\
M_\theta &= \frac{P}{4\sin 2\varphi}\left[\mu M_3(\xi) + (1-\mu)\overline{M}_3(\xi)\right],\\
Q_\rho &= -\frac{P}{4L_0\sin 2\varphi}Q_3(\xi),
\end{aligned}\right\} \tag{5.10}$$

The functions θ_3, M_3, $\overline{M}_3$, Q_3 are given by (2.25) through (2.28), or, in expanded form, by the following series:

$$\left.\begin{aligned}
\theta_3(\xi) &= \left(1-\frac{2\varphi}{\pi}\right)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,(m+1)!}\left(\frac{\xi}{2}\right)^{2m+1}\cos 2(m+1)\varphi -\\
&\quad -\frac{2}{\pi}\left(\ln\frac{\xi}{2}+c\right)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,(m+1)!}\left(\frac{\xi}{2}\right)^{2m+1}\sin 2(m+1)\varphi +\\
&\quad +\frac{\xi}{2\pi}\sin 2\varphi + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{m!\,(m+1)!}\times\\
&\quad \times\left(\frac{\xi}{2}\right)^{2m+1}\sin 2(m+1)\varphi\left(1+\frac{1}{2}+\ldots+\frac{1}{m}\right),\\
M_3(\xi) &= \left(1-\frac{2\varphi}{\pi}\right)\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{\xi}{2}\right)^{2m}\sin 2(m+1)\varphi -\\
&\quad -\frac{2}{\pi}\left(\ln\frac{\xi}{2}+c\right)\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{\xi}{2}\right)^{2m}\sin 2(m+1)\varphi -\\
&\quad -\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{\xi}{2}\right)^{2m}\sin 2(m+1)\varphi\left(1+\frac{1}{2}+\right.\\
&\qquad\left.+\frac{1}{3}+\ldots+\frac{1}{m}\right),\\
\overline{M}_3(\xi) &= \left(1-\frac{2\varphi}{\pi}\right)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,(m+1)!}\left(\frac{\xi}{2}\right)^{2m}\cos 2(m+1)\varphi -\\
&\quad -\frac{2}{\pi}\left(\ln\frac{\xi}{2}+c\right)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,(m+1)!}\left(\frac{\xi}{2}\right)^{2m}\sin 2(m+1)\varphi +\\
&\quad +\frac{\sin 2\varphi}{\pi}+\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{m!\,(m+1)!}\times\\
&\quad \times\left(\frac{\xi}{2}\right)^{2m}\sin 2(m+1)\varphi\left(1+\frac{1}{2}+\ldots+\frac{1}{m}\right),\\
Q_3(\xi) &= \left(1-\frac{2\varphi}{\pi}\right)\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{\xi}{2}\right)^{2m+1}\cos 2(m+2)\varphi -\\
&\quad -\frac{2}{\pi}\left(\ln\frac{\xi}{2}+c\right)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,(m+1)!}\left(\frac{\xi}{2}\right)^{2m+1}\sin 2(m+2)\varphi -\\
&\quad -\frac{2}{\pi\xi}\sin 2\varphi - \frac{\xi}{2\pi}\sin 4\varphi + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{m!\,(m+1)!}\times\\
&\quad \times\left(\frac{\xi}{2}\right)^{2m+1}\sin 2(m+2)\varphi\left(1+\frac{1}{2}+\ldots+\frac{1}{m}\right).
\end{aligned}\right\} \tag{5.11}$$

Taking a finite number of terms in (5.11), it is possible to determine, with sufficient accuracy, the displacements, moments, and forces in the most highly loaded sections (for $\rho \to 0$).

At $\varphi = 45°$, i.e., if no shearing forces act in the elastic foundation, the solution obtained coincides with Hertz's solution for an infinite plate on an elastic Winkler foundation.

§ 6. ELASTIC PLATE OF FINITE DIMENSIONS

The general solution for a circular plate on an elastic foundation was given in section 2 of this chapter. We shall now give some examples which clarify the general theory.

For practical calculations, tables of Bessel functions of the first kind, of a complex argument, and tables of the functions:

$$\theta_1(\xi),\ \theta_2(\xi),\ M_1(\xi),\ M_2(\xi),\ \overline{M}_1(\xi),\ \overline{M}_2(\xi),$$

which define the states of strain and stress of the plate, are given in Table 12 of the appendix. The tables have been compiled for $45° \leqslant \varphi \leqslant 65°$, and give the various functions for:

$$\xi = \frac{\rho}{L_0} = 0;\ 0{,}05;\ 0{,}10;\ 0{,}15;\ \ldots;\ 1{,}0.$$

When the dimensions and physical properties of the plates are such that the argument ξ of the corresponding functions is not contained in the tables, reference has to be made to /4/ and /86/. After the Bessel functions have been determined, the functions:

$$\theta_1(\xi),\ \theta_2(\xi),\ \ldots,\ \overline{M}_2(\xi),$$

are obtained from (2.25) through (2.28).

In addition, series defining all the required magnitudes are given in the examples below in order to permit calculations for:

$$\xi_R = \frac{R}{L_0} > 1{,}0$$

without having recourse to tables.

1. Circular plate under the action of a uniformly distributed load.

Consider a circular plate of radius R, subjected to a uniformly distributed load p, lying on an elastic single-layer foundation (Figure 119).

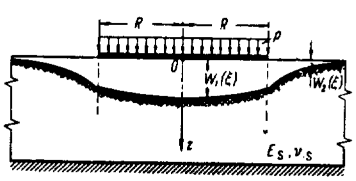


FIGURE 119.

The differential equation of bending is:

$$\nabla_\xi^2\nabla_\xi^2 W_1 - 2r_0^2\nabla_\xi^2 W_1 + W_1 = \frac{pL_0^4}{D}, \tag{6.1}$$

where $\xi = \frac{\rho}{L_0}$, and

$$r_0^2 = \frac{tL_0^2}{D}, \quad L_0 = \sqrt[4]{\frac{D}{k}}. \tag{6.2}$$

The following homogeneous differential equation holds true beyond the plate edges:

$$\nabla_\xi^2 W_2 - \alpha_0^2 W_2 = 0, \tag{6.3}$$

where

$$\alpha_0^2 = \alpha^2 L_0^2 = \frac{kL_0^2}{2t} \tag{6.4}$$

and

$$\nabla_\xi^2 = \frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi}. \tag{6.5}$$

The general solution of (6.1) and (6.3) is:

$$W_1 = C_1 u_0(\xi) + C_2 v_0(\xi) + C_3 f_0(\xi) + C_4 g_0(\xi) + \frac{p}{k}, \tag{6.6}$$

$$W_2 = C_5 I_0(\alpha_0\xi) + C_6 K_0(\alpha_0\xi), \tag{6.7}$$

where $\frac{p}{k}$ = particular integral of the nonhomogeneous differential equation (6.1); $I_0(\alpha_0\xi)$, $K_0(\alpha_0\xi)$ = modified zero-order Bessel functions of the first and second kind, of the argument $\alpha_0\xi$; $C_1, \ldots, C_6$ = integration constants.

To determine these six integration constants, the following six independent boundary conditions are used:

at $\rho = 0 \quad (\xi = 0)$: $$\frac{dW_1}{d\rho} = 0, \quad \int_0^{2\pi} Q_\rho \rho \, d\theta = 0; \tag{6.8}$$

at $\rho = R \quad \left(\xi = \frac{R}{L_0}\right)$: $$\left.\begin{aligned} &M_\rho = 0, \\ &Q_\rho = 2t\left(\frac{dW_2}{d\rho} - \frac{dW_1}{d\rho}\right), \quad W_1(R) = W_2(R); \end{aligned}\right\} \tag{6.9}$$

at $\rho \to \infty \quad (\xi \to \infty)$: $$W_2(\rho) = 0. \tag{6.10}$$

Conditions (6.8) state that the slope and the shearing force vanish at the plate center; condition (6.10), states that the vertical displacements of the elastic foundation vanish at infinity. Expressions (6.9) are the boundary conditions at the free plate edge $\rho = R$; the second condition (6.9) accounts for the effect of the free foundation beyond the plate edges on the stresses in the plate (cf. (1.8)).

Recalling the considerations which led to (3.4), (5.4), [and (5.9)], we obtain:

$$C_3 = C_4 = C_5 = 0, \tag{6.11}$$

The plate deflections W_1 and the vertical displacements W_2 of the free foundation surface are then:

$$W_1 = C_1 u_0(\xi) + C_2 v_0(\xi) + \frac{p}{k}. \tag{6.12}$$

$$W_2 = C_6 K_0(\alpha_0 \xi). \tag{6.13}$$

From the third condition (6.9) we obtain:

$$C_6 = \frac{C_1 u_0(\xi_R) + C_2 v_0(\xi_R) + \frac{p}{k}}{K_0(\alpha_0 \xi_R)}, \tag{6.14}$$

where $\xi_R = \frac{R}{L_0}$.

The remaining two constants C_1 and C_2 can be determined from the first two conditions (6.9), rewritten with the aid of (1.4) in the following form:

$$\left.\begin{aligned} \nabla_\xi^2 W_1 - \frac{1-\mu}{\xi}\frac{dW_1}{d\xi} &= 0, \\ \frac{d}{d\xi}\nabla_\xi^2 W_1 &= -\frac{2tL_0^2}{D}\left[\frac{dW_2}{d\xi} - \frac{dW_1}{d\xi}\right]. \end{aligned}\right\} \tag{6.15}$$

Substituting (6.12) and (6.13) in (6.15) and using the rules of differentiation of cylindrical functions, we obtain:

$$\left.\begin{aligned} m_1 C_1 + m_2 C_2 &= 0, \\ n_1 C_1 + n_2 C_2 &= G_p, \end{aligned}\right\} \tag{6.16}$$

where:

$$\left.\begin{aligned} m_1 &= M_1(\xi_R) - (1-\mu)\overline{M}_1(\xi_R), \\ m_2 &= M_2(\xi_R) - (1-\mu)\overline{M}_2(\xi_R), \\ n_1 &= Q_1(\xi_R) + \frac{\theta_1(\xi_R)}{\alpha_0^2} - u_0(\xi_R)\frac{K_1(\alpha_0\xi_R)}{\alpha_0 K_0(\alpha_0\xi_R)}, \\ n_2 &= Q_2(\xi_R) + \frac{\theta_2(\xi_R)}{\alpha_0^2} - v_0(\xi_R)\frac{K_1(\alpha_0\xi_R)}{\alpha_0 K_0(\alpha_0\xi_R)}, \\ G_p &= \frac{p}{k}\frac{K_1(\alpha_0\xi_R)}{\alpha_0 K_0(\alpha_0 \xi_R)}. \end{aligned}\right\} \tag{6.17}$$

The solution of system (6.16) is:

$$\left.\begin{aligned} C_1 &= -\frac{G_p m_2}{m_1 n_2 - m_2 n_1}, \\ C_2 &= \frac{G_p m_1}{m_1 n_2 - m_2 n_1}. \end{aligned}\right\} \tag{6.18}$$

The displacements, moments, and forces in the circular plate are then:

$$\left.\begin{aligned}
W_1 &= C_1 u_0(\xi) + C_2 v_0(\xi) + \frac{p}{R},\\
\frac{dW_1}{d\rho} &= -\frac{1}{L_0}[C_1\theta_1(\xi) + C_2\theta_2(\xi)],\\
M_\rho &= \frac{D}{L_0^2}[C_1 L_1(\xi) + C_2 L_2(\xi)],\\
M_\theta &= \frac{D}{L_0^2}[C_1\overline{L}_1(\xi) + C_2\overline{L}_2(\xi)],\\
Q_\rho &= -\frac{D}{L_0^3}[C_1 Q_1(\xi) + C_2 Q_2(\xi)].
\end{aligned}\right\} \tag{6.19}$$

where

$$\left.\begin{aligned}
u_0(\xi) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{\xi}{2}\right)^{2m} \cos 2m\varphi,\\
v_0(\xi) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{\xi}{2}\right)^{2m} \sin 2m\varphi,\\
\theta_1(\xi) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)(m+1)!}\left(\frac{\xi}{2}\right)^{2m+1} \cos 2(m+1)\varphi,\\
\theta_2(\xi) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!}\left(\frac{\xi}{2}\right)^{2m+1} \sin 2(m+1)\varphi,\\
M_1(\xi) &= \sum_{m=0}^{\infty} \frac{(-1)_m}{(m!)^2}\left(\frac{\xi}{2}\right)^{2m} \cos 2(m+1)\varphi,\\
M_2(\xi) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{\xi}{2}\right)^{2m} \sin 2(m+1)\varphi,\\
\overline{M}_1(\xi) &= \frac{1}{2}\sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!}\left(\frac{\xi}{2}\right)^{2m} \cos 2(m+1)\varphi,\\
\overline{M}_2(\xi) &= \frac{1}{2}\sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!}\left(\frac{\xi}{2}\right)^{2m} \sin 2(m+1)\varphi,\\
Q_1(\xi) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!}\left(\frac{\xi}{2}\right)^{2m+1} \cos 2(m+2)\varphi,\\
Q_2(\xi) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!}\left(\frac{\xi}{2}\right)^{2m+1} \sin 2(m+2)\varphi,\\
L_1(\xi) &= M_1(\xi) - (1-\mu)\overline{M}_1(\xi),\\
L_2(\xi) &= M_2(\xi) - (1-\mu)\overline{M}_2(\xi),\\
\overline{L}_1(\xi) &= \mu M_1(\xi) + (1-\mu)\overline{M}_1(\xi),\\
\overline{L}_2(\xi) &= \mu M_2(\xi) + (1-\mu)\overline{M}_2(\xi).
\end{aligned}\right\} \tag{6.20}$$

2. Circular plate under the action of a uniformly distributed edge load

If a circular plate is subjected to the action of an edge load P_k(Figure 120), expression (6.1) reduces to a homogeneous differential equation. The general solution of the problem considered is therefore:

$$\left.\begin{aligned}
W_1 &= C_1 u_0(\xi) + C_2 v_0(\xi) + C_3 f_0(\xi) + C_4 g_0(\xi),\\
W_2 &= C_5 I_0(\alpha_0\xi) + C_6 K_0(\alpha_0\xi).
\end{aligned}\right\} \tag{6.21}$$

The boundary conditions are given, as in the first example, by (6.8) through (6.10), only the second condition (6.9) being replaced by:

$$\rho = R \qquad Q_\rho = 2t\left(\frac{dW_2}{d\rho} - \frac{dW_1}{d\rho}\right) + P_k. \tag{6.22}$$

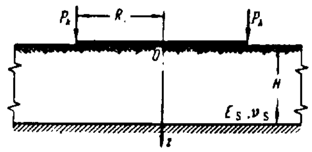


FIGURE 120.

By analogy with (6.11) and (6.14), we obtain:

$$C_3 = C_4 = C_5 = 0, \qquad C_6 = \frac{C_1 u_0(\xi_R) + C_2 v_0(\xi_R)}{K_0(\alpha_0 \xi_R)}. \tag{6.23}$$

The plate deflections W_1 and the vertical displacements W_2 of the free foundation surface ($\rho > R$) are then:

$$W_1 = C_1 u_0(\xi) + C_2 v_0(\xi), \qquad W_2 = C_6 K_0(\alpha_0 \xi). \tag{6.24}$$

From (6.22) and the first condition (6.9), we obtain:

$$\left.\begin{aligned} &\nabla_\xi^2 W_1 - \frac{1-\mu}{\xi}\frac{dW_1}{d\xi} = 0, \\ &\frac{d}{d\xi}\nabla_\xi^2 W_1 + \frac{2tL_0^2}{D}\left(\frac{dW_2}{d\xi} - \frac{dW_1}{d\xi}\right) + \frac{P_k L_0^3}{D} = 0. \end{aligned}\right\} \tag{6.25}$$

Substitution of (6.24) in (6.25) leads to a system of two algebraic equations in the constants C_1 and C_2:

$$m_1 C_1 + m_2 C_2 = 0, \qquad n_1 C_1 + n_2 C_2 = G_{P_k}, \tag{6.26}$$

where m_1, m_2, n_1, n_2 are determined from (6.17), while

$$G_{P_k} = -\frac{P_k L_0^3}{D}. \tag{6.27}$$

The integration constants C_1 and C_2 are thus again given by (6.18), with G_p replaced by G_{P_k}. The plate deflections are determined from the first equation (6.24), while the slopes, bending moments, and shearing forces are given by (6.19).

3. Circular plate under the action of moments distributed along the edge

Consider the plate shown in Figure 121. It can be seen that all the results obtained before also apply in this case. In fact, the general solution for $0 \leqslant \xi \leqslant \xi_R$ and $\xi_R \leqslant \xi < \infty$ can be written in the form (6.21), whence, as above:

$$\left.\begin{aligned} &C_3 = C_4 = C_5 = 0, \\ &C_6 = \frac{C_1 u_0(\xi_R) + C_2 v_0(\xi_R)}{K_0(\alpha_0 \xi_R)}. \end{aligned}\right\} \tag{6.28}$$

The integrations constants C_1 and C_2 are determined from the boundary conditions:

$$\rho = R \ (\xi = \xi_R) \quad M = M_k, \quad Q = 2t\left(\frac{dW_2}{d\rho} - \frac{dW_1}{d\rho}\right). \tag{6.29}$$

From (1.4) and (6.21), taking (6.28) into account, we obtain:

$$m_1 C_1 + m_2 C_2 = G_M, \quad n_1 C_1 + n_2 C_2 = 0. \tag{6.30}$$

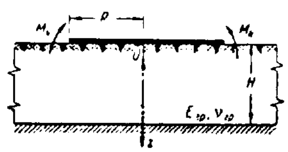


Рис. 121.

The coefficients in (6.3) are obtained, as before, from (6.17), while:

$$G_M = -\frac{M_k L_0^2}{D}. \tag{6.31}$$

Thus, the constants C_1 and C_2 are:

$$C_1 = \frac{m_2 G_M}{m_1 n_2 - n_1 m_2}, \quad C_2 = -\frac{n_1 G_M}{m_1 n_2 - n_1 m_2}. \tag{6.32}$$

Chapter V

AXISYMMETRICAL DEFORMATION OF A SHALLOW SPHERICAL SHELL ON A SINGLE-LAYER ELASTIC FOUNDATION

§ 1. BASIC DIFFERENTIAL EQUATIONS OF THE THEORY OF SHALLOW SPHERICAL SHELLS

1

A shallow shell is a thin-walled three-dimensional structure whose height is small in comparison with its dimensions in plan. A shell is called shallow if the ratio of its height to its smallest dimension in plan l_{min} is less than $^1/_5$.

Since for $\frac{f}{l_{min}} < \frac{1}{5}$ the shell curvatures are very small, we can apply Euclidian plane geometry to the middle surface of a shallow shell. This assumption is equivalent to replacing the first fundamental form [also known as ground form] of the shallow-shell surface by the corresponding fundamental form for a plane. This also means that the Gauss curvature:

$$K = k_1 k_2 = \frac{1}{R_1 R_2},$$

is very small for shallow shells and can be approximated to zero.

An additional assumption, made when considering the general equations of equilibrium of a shallow shell, is that only the principal moment terms, which do not contain as factors surface curvatures and curvature derivatives, need be taken into account. All other moment terms are neglected as being very small and having an insignificant effect on the internal forces and bending moments of the shell*.

2

Consider a shallow shell having the form of a spherical surface of radius R (Figure 122). Let ρ and θ be the polar coordinates measured in the plane of the shell base. It will be assumed that the projection of the shell apex on this plane coincides with the origin of the ρ, θ coordinate system. On the strength of the geometrical hypotheses underlying the general theory of shallow shells, all points of the middle surface of the shell will be defined by the coordinates ρ and θ.

* The general theory of shallow shells was first expounded by V. Z. Vlasov. For a detailed treatment of this theory, see /8/.

Let the shell considered be subjected only to the action of a normal load Z, positive when directed along the outer normal (Figure 122). In this case all statical and geometrical equations characterizing the states of stress and strain of the shallow spherical shell can be reduced to the following system of two differential equations:

$$\left.\begin{aligned} &\frac{R}{Eh}\nabla^2\nabla^2\Phi - \nabla^2 w = 0,\\ &\frac{1}{R}\nabla^2\Phi + D\nabla^2\nabla^2 w - Z = 0, \end{aligned}\right\} \qquad (1.1)$$

where $w = w(\rho, \theta)$ = radial displacement of shell (positive if directed along the outer normal), and $\Phi = \Phi(\rho, \theta)$ = stress function determining membrane forces acting in shell:

$$\left.\begin{aligned} N_\rho &= \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\theta^2},\\ N_\theta &= \frac{\partial^2\Phi}{\partial\rho^2},\\ S &= -\frac{1}{\rho}\frac{\partial^2\Phi}{\partial\rho\,\partial\theta} + \frac{1}{\rho^2}\frac{\partial\Phi}{\partial\theta}. \end{aligned}\right\} \qquad (1.2)$$

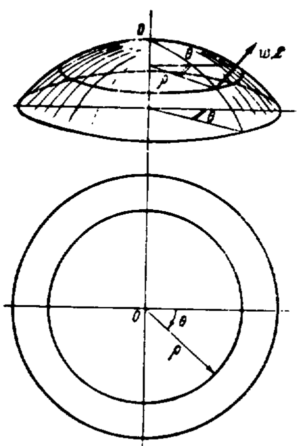


FIGURE 122.

The symbol ∇^2 in (1.1) denotes the second-order differential operator:

$$\nabla^2 = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho}\frac{\partial^2}{\partial\theta^2}\right]. \qquad (1.3)$$

The magnitudes h and D represent the shell thickness and its flexural rigidity respectively:

$$D = \frac{Eh}{12(1-\mu^2)}. \qquad (1.4)$$

The first equation (1.1) has a geometrical meaning: it expresses the condition of continuity of the deformations; the second equation has a statical meaning: it characterizes the equilibrium condition of the shell in the radial direction.

We introduce the scalar function $F = F(\rho, \theta)$, which satisfies the following relationships:

$$\left.\begin{aligned} w &= \nabla^2\nabla^2 F, \\ \Phi &= \frac{Eh}{R}\nabla^2 F, \end{aligned}\right\} \tag{1.5}$$

The first equation (1.1) is then satisfied identically. Substituting (1.5) in the second equation (1.1) yields:

$$D\nabla^2\nabla^2\nabla^2\nabla^2 F + \frac{Eh}{R^2}\nabla^2\nabla^2 F - Z = 0. \tag{1.6}$$

or

$$D\nabla^2\nabla^2 w + \frac{Eh}{R^2} w - Z = 0. \tag{1.7}$$

The radial displacements w of a spherical shell are thus determined by a fourth-order differential equation having the same form as the equation of bending of a plate on an elastic Winkler foundation whose foundation modulus is

$$k = \frac{Eh}{R^2}.$$

Hence, with respect to the strains due to the deflections w, complete analogy exists between a shallow spherical shell and a circular plate on an elastic foundation, suitably supported along the edge. Exactly as in the case of bending of a circular plate, the additional curvatures $\varkappa_\rho, \varkappa_\theta$ and the twist τ are:

$$\left.\begin{aligned} \varkappa_\rho &= \frac{\partial^2 w}{\partial \rho^2}, \\ \varkappa_\theta &= \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{\rho}\frac{\partial w}{\partial \rho}, \\ \tau &= \frac{1}{\rho}\frac{\partial^2 w}{\partial \rho\, \partial \theta} - \frac{1}{\rho^2}\frac{\partial w}{\partial \theta}. \end{aligned}\right\} \tag{1.8}$$

3

The internal forces and moments acting in sections $\rho = \text{const}$, $\theta = \text{const}$, of a shallow spherical shell can be divided into two groups: the normal (membrane) forces N_ρ, N_θ, S, corresponding to deformation without bending, and the bending moments M_ρ, M_θ, torques H, and shearing forces Q_ρ, Q_θ due to bending. The positive directions of these forces and moments for surfaces with positive outer normals are shown in Figure 123.

As already mentioned, the first group of forces is determined by the stress function $\Phi = \Phi(\rho, \theta)$ with the aid of (1.2). The second group is determined by the displacement function $w = w(\rho, \theta)$. By virtue of the analogy noted above, these forces and moments are calculated in the same way as in the case of bending of a circular plate:

$$\left.\begin{aligned} M_\rho &= -D(\varkappa_\rho + \mu\varkappa_\theta), \\ M_\theta &= -D(\varkappa_\theta + \mu\varkappa_\rho), \\ H &= D(1-\mu)\tau, \end{aligned}\right\} \tag{1.9}$$

$$\left.\begin{aligned} Q_\rho &= -D\frac{\partial}{\partial\rho}\nabla^2 w, \\ Q_\theta &= -D\frac{1}{\rho}\frac{\partial}{\partial\theta}\nabla^2 w, \end{aligned}\right\} \tag{1.10}$$

where $\varkappa_\rho, \varkappa_\theta, \tau$ are defined by (1.8), and ∇^2 by (1.3).

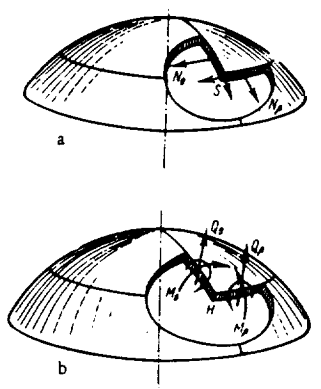


FIGURE 123.

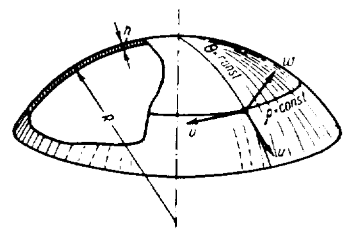


FIGURE 124.

We determine the tangential displacements $u(\rho,\theta)$ and $v(\rho,\theta)$ in the directions of the tangents to the curves $\rho = \text{const}$, $\theta = \text{const}$ respectively (Figure 124), from the relationships between the membrane forces N_ρ, N_θ, S and the strains:

$$N_\rho = \frac{Eh}{(1-\mu^2)}(\varepsilon_\rho + \mu\varepsilon_\theta), \quad N_\theta = \frac{Eh}{(1-\mu^2)}(\varepsilon \ + \mu\varepsilon_\rho), \quad S = \frac{Eh}{2(1+\mu)}\omega, \tag{1.11}$$

where $\varepsilon_\rho, \varepsilon_\theta, \omega$ are the tensile (compressive) and shearing strain respectively, determined by the displacements u, v, and w as follows:

$$\left.\begin{aligned} \varepsilon_\rho &= \frac{\partial u}{\partial\rho} + \frac{w}{R}, \\ \varepsilon_\theta &= \frac{1}{\rho}\frac{\partial v}{\partial\theta} + \frac{u}{\rho} + \frac{w}{R}, \\ \omega &= \frac{1}{\rho}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial\rho} - \frac{v}{R}. \end{aligned}\right\} \tag{1.12}$$

When the stress function Φ and the displacement function w have been determined from (1.1), and then $\varepsilon_\rho, \varepsilon_\theta$, and ω from (1.2) and (1.11), u and v can be found from (1.12).

§ 2. DIFFERENTIAL EQUATIONS OF A SPHERICAL SHELL ON AN ELASTIC SINGLE-LAYER FOUNDATION

Let a shallow spherical shell lie on an elastic foundation whose properties are described by the differential equation:

$$2t\nabla^2 w - kw + q = 0, \tag{2.1}$$

where ∇^2 is defined by (1.3), k and t are the generalized characteristics of the elastic foundation, and q is the load per unit area, acting on the foundation.

Since the radial displacements w of the shell and of the elastic foundation are equal along the entire area of contact (Figure 125), (1.7) and (2.1) can be considered simultaneously:

$$D\nabla^2\nabla^2 w + \frac{Eh}{R^2} w - Z = 0, \quad 2t\nabla^2 w - kw + q = 0. \tag{2.2}$$

The external load on the shell consists of the known forces p and the foundation reactions q [all referred to unit area]:

$$Z = p - q; \tag{2.3}$$

eliminating q from (2.2), we obtain:

$$D\nabla^2\nabla^2 w - 2t\nabla^2 w + \left(k + \frac{Eh}{R^2}\right) w = p. \tag{2.4}$$

This equation has the same form as the equation of bending of a thin plate on a single-layer foundation (cf. for instance (1.1) of Chapter IV), differing from it only in the coefficient of w. This coefficient is larger by Eh/R^2 than the corresponding coefficient in (1.1) of Chapter IV.

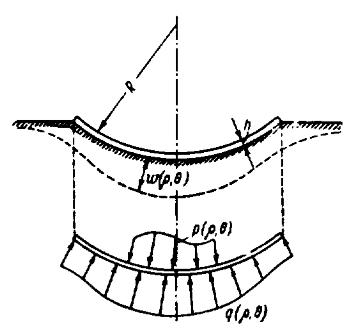


FIGURE 125.

The problem of bending of a spherical shell on a single-layer elastic foundation is thus similar to the corresponding problem of a circular plate, discussed in Chapter IV.

To determine the stresses and displacements when the membrane problem is considered (in which bending stresses are neglected), we use the first equation (1.1):

$$\frac{R}{Eh}\nabla^2\nabla^2\Phi = \nabla^2 w. \tag{2.5}$$

This is a nonhomogeneous biharmonic equation, the function w being assumed known (it can be determined from (2.4)). Having determined the function Φ from (2.5) and the corresponding boundary conditions, the normal forces, moments, strains, and displacements of the spherical shell can be found from (1.2), (1.11), (1.12).

If the problem is axisymmetrical, all derivatives with respect to θ vanish in the equations determining the states of stress and strain of the shell. The Laplacian then reduces to:

$$\nabla^2 = \frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho}, \tag{2.6}$$

while (1.9) and (1.10) become (taking (1.8) into account):

$$\left.\begin{aligned} M_\rho &= -D\left[\nabla^2 W - \frac{(1-\mu)}{\rho}\frac{dW}{d\rho}\right],\\ M_\theta &= -D\left[\mu\nabla^2 W + \frac{(1-\mu)}{\rho}\frac{dW}{d\rho}\right],\\ Q_\rho &= -D\frac{d}{d\rho}\nabla^2 W,\\ Q_\theta &= H = 0. \end{aligned}\right\} \tag{2.7}$$

Equation (1.2) and (1.12) respectively take the form:

$$N_\rho = \frac{1}{\rho}\frac{d\Phi}{d\rho}, \qquad N_\theta = \frac{d^2\Phi}{d\rho^2}, \qquad S = 0; \tag{2.8}$$

$$\varepsilon_\rho = \frac{du}{d\rho} + \frac{W}{R}, \qquad \varepsilon_\theta = \frac{u}{\rho} + \frac{W}{R} \qquad \omega = 0, \tag{2.9}$$

where $W = W(\rho)$, $\Phi = \Phi(\rho)$, depend on ρ only.

§ 3. GENERAL SOLUTION FOR AXISYMMETRICAL DEFORMATIONS

1. Bending of shells

Exactly as in bending of a plate, we replace ρ by the dimensionless coordinate $\xi = \frac{\rho}{L_0}$, where

$$L_0 = \sqrt[4]{\frac{DR^2}{Eh + kR^2}} \tag{3.1}$$

Equation (2.4) then becomes:

$$\nabla_\xi^2\nabla_\xi^2 W - 2r^2\nabla_\xi^2 W + W = \frac{pL_0^4}{D}, \tag{3.2}$$

where

$$r^2 = \frac{tL_0^2}{D}, \tag{3.3}$$

and

$$\nabla_\xi^2 = \frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi} \qquad (3.4)$$

By analogy with the integral of (2.2) of Chapter IV, the general integral of (3.2) can be presented in the form:

$$W = C_1 u_0(\xi) + C_2 v_0(\xi) + C_3 f_0(\xi) + C_4 g_0(\xi) + W_p, \qquad (3.5)$$

where

$$u_0(\xi),\ v_0(\xi),\ f_0(\xi),\ g_0(\xi)$$

are respectively the real and imaginary parts of the zero-order Bessel and Hankel functions, and W_p is a particular integral of (3.2). On the basis of the general solution of (3.2), all statical and kinematic magnitudes referring to the state of bending stress of the shell are determined from (2.24) of Chapter IV, with L_0 defined by (3.1).

2. Deformation without bending of shells

We determine the stress function $\Phi = \Phi(\rho)$ with the aid of (2.5). Replacing the variable ρ by the dimensionless coordinate $\xi = \rho / L_0$, where L_0 is defined by (3.1), (2.5) becomes:

$$\nabla_\xi^2 \nabla_\xi^2 \Phi = \frac{EhL_0^2}{R} \nabla_\xi^2 W. \qquad (3.6)$$

The function $W = W(\xi)$ is defined by (3.5), and the Laplacian by (3.4). The general solution of (3.6) is then:

$$\Phi = \frac{EhL_0^2}{R}(\Phi_0 + \Phi_W), \qquad (3.7)$$

where $\Phi_0 = \Phi_0(\xi)$ is the general solution of the homogeneous biharmonic equation corresponding to (3.6) and $\Phi_W = \Phi_W(\xi)$ is a particular integral of the nonhomogeneous equation:

$$\nabla_\xi^2 \nabla_\xi^2 \Phi_W = \nabla_\xi^2 W. \qquad (3.8)$$

The solution of the homogeneous biharmonic equation is for the axi-symmetrical case is:

$$\Phi_0 = C_5 + C_6\xi^2 + C_7\xi^2 \ln\xi + C_8 \ln\xi, \qquad (3.9)$$

where C_5, C_6, C_7, C_8 are constants.

The particular integral of the nonhomogeneous biharmonic equation (3.8) is:

$$\Phi_W = -[C_1\varphi_1(\xi) + C_2\varphi_2(\xi) + C_3\varphi_3(\xi) + C_4\varphi_4(\xi)] + \Phi_p, \qquad (3.10)$$

where Φ_p is a particular integral corresponding to the particular integral W_p of (3.5) and:

$$\left.\begin{aligned}\varphi_1(\xi) &= u_0(\xi)\cos 2\varphi + v_0(\xi)\sin 2\varphi,\\ \varphi_2(\xi) &= -u_0(\xi)\sin 2\varphi + v_0(\xi)\cos 2\varphi,\\ \varphi_3(\xi) &= f_0(\xi)\cos 2\varphi + g_0(\xi)\sin 2\varphi,\\ \varphi_4(\xi) &= -f_0(\xi)\sin 2\varphi + g_0(\xi)\cos 2\varphi,\end{aligned}\right\} \tag{3.11}$$

Here as before:

$$\varphi = \arg(\sqrt{a}) = \frac{1}{2}\arg a,$$

where a is a complex number defined by (2.11) of Chapter IV in accordance with expressions (3.3) and (3.1) of this chapter.

It is easily seen by direct substitution that (3.10) does in fact satisfy the nonhomogeneous equation (3.8).

In accordance with (3.9) and (3.10), the general integral of (3.6) is therefore:

$$\begin{aligned}\Phi = \frac{EhL_0^2}{R}[&-C_1\varphi_1(\xi) - C_2\varphi_2(\xi) - C_3\varphi_3(\xi) - C_4\varphi_4(\xi) +\\ &+ C_5 + C_6\xi^2 + C_7\xi^2\ln\xi + C_8\ln\xi + \Phi_p].\end{aligned} \tag{3.12}$$

Substituting this in (2.8) yields the following expressions for the normal forces:

$$\begin{aligned}N_\rho = \frac{Eh}{R}\Big[&C_1n_1(\xi) + C_2n_2(\xi) + C_3n_3(\xi) + C_4n_4(\xi) +\\ &+ 2C_6 + C_7(1 + 2\ln\xi) + C_8\frac{1}{\xi^2} + \frac{1}{\xi}\frac{d\Phi_p}{d\xi}\Big],\end{aligned} \tag{3.13}$$

$$\begin{aligned}N_\theta = \frac{Eh}{R}\Big\{&C_1[u_0(\xi) - n_1(\xi)] + C_2[v_0(\xi) - n_2(\xi)] +\\ &+ C_3[f_0(\xi) - n_3(\xi)] + C_4[g_0(\xi) - n_4(\xi)] + 2C_6 +\\ &+ C_7(3 + 2\ln\xi) - C_8\frac{1}{\xi^2} + \frac{d^2\Phi_p}{d\xi^2}\Big\},\end{aligned} \tag{3.14}$$

where

$$\left.\begin{aligned}n_1(\xi) &= \frac{1}{\xi}[u_1(\xi)\cos\varphi + v_1(\xi)\sin\varphi],\\ n_2(\xi) &= \frac{1}{\xi}[-u_1(\xi)\sin\varphi + v_1(\xi)\cos\varphi],\\ n_3(\xi) &= \frac{1}{\xi}[f_1(\xi)\cos\varphi + g_1(\xi)\sin\varphi],\\ n_4(\xi) &= \frac{1}{\xi}[-f_1(\xi)\sin\varphi + g_1(\xi)\cos\varphi],\end{aligned}\right\} \tag{3.15}$$

and u_1, v_1, f_1, g_1 are respectively the real and imaginary parts of the first-order Bessel and Hankel functions.

To obtain the deformations of the shell, we rewrite (2.9) in dimensionless coordinates:

$$\left.\begin{aligned}\varepsilon_\rho &= \frac{1}{L_0}\frac{du}{d\xi} + \frac{W}{R},\\ \varepsilon_\theta &= \frac{1}{L_0}\frac{u}{\xi} + \frac{W}{R}.\end{aligned}\right\} \tag{3.16}$$

From these expressions and from (1.11) we obtain:

$$N_\rho + N_\theta = \frac{Eh}{1-\mu}(\varepsilon_\rho + \varepsilon_\theta) = \frac{Eh}{1-\mu}\left[\frac{1}{L_0}\left(\frac{du}{d\xi} + \frac{u}{\xi}\right) + \frac{2W}{R}\right]. \tag{3.17}$$

On the other hand, (2.8), written in dimensionless coordinates, and (3.7) lead to:

$$N_\rho + N_\theta = \frac{1}{L_0^2}\nabla_\xi^2\Phi = \frac{Eh}{R}\nabla_\xi^2(\Phi_0 + \Phi_W). \tag{3.18}$$

Since, by (3.8):

$$\nabla_\xi^2\Phi_W = W, \tag{3.19}$$

we can rewrite (3.18) in the form:

$$N_\rho + N_\theta = \frac{Eh}{R}(\nabla_\xi^2\Phi_0 + W). \tag{3.20}$$

Equating the right sides of (3.17) and (3.20) yields:

$$\frac{Eh}{R}[\nabla_\xi^2\Phi_0 + W] = \frac{Eh}{1-\mu}\left[\frac{1}{L_0}\left(\frac{du}{d\xi} + \frac{u}{\xi}\right) + \frac{2W}{R}\right], \tag{3.21}$$

or

$$\frac{du}{d\xi} + \frac{u}{\xi} = \frac{L_0}{R}[(1-\mu)\nabla_\xi^2\Phi_0 - (1+\mu)W]. \tag{3.22}$$

This equation establishes the relationship between the unknown tangential displacement u and the known functions Φ_0 and W. Substitution of (3.9) and (3.5) in (3.22) yields finally:

$$\begin{aligned}\frac{du}{d\xi} + \frac{u}{\xi} = \frac{L_0}{R}\{(1-\mu)[4C_6 + 4C_7(\ln\xi + 1)] - \\ -(1+\mu)[C_1u_0(\xi) + C_2v_0(\xi) + C_3f_0(\xi) + C_4g_0(\xi) + W_p]\}.\end{aligned} \tag{3.23}$$

The general solution of (3.23) is:

$$\begin{aligned}u = \frac{L_0}{R}\Big\{-(1+\mu)\Big[C_1\chi_1(\xi) + C_2\chi_2(\xi) + C_3\chi_3(\xi) + C_4\chi_4(\xi) + \\ + A_1\frac{1}{\xi}\Big] + (1-\mu)[2C_6\xi + C_7(2\xi\ln\xi + \xi)] + u_p\Big\},\end{aligned} \tag{3.24}$$

where

$$A_1\frac{L_0(1+\mu)}{R}\frac{1}{\xi}$$

is the integral of the homogeneous equation corresponding to (3.23), determined up to the constant factor $\frac{L_0(1+\mu)}{R}$; u_p is a particular integral corresponding to the function W_p; and:

$$\left.\begin{aligned}\chi_1(\xi) &= u_1(\xi)\cos\varphi + v_1(\xi)\sin\varphi,\\ \chi_2(\xi) &= -u_1(\xi)\sin\varphi + v_1(\xi)\cos\varphi,\\ \chi_3(\xi) &= f_1(\xi)\cos\varphi + g_1(\xi)\sin\varphi,\\ \chi_4(\xi) &= -f_1(\xi)\sin\varphi + g_1(\xi)\cos\varphi.\end{aligned}\right\} \tag{3.25}$$

Due to the summation of (3.13) and (3.14), the constant C_8 no longer appears in (3.23). On the other hand, a new constant A_1 appears in (3.24) It is seen that these two constants are identical by substituting the solution:

$$u_0 = -\frac{L_0(1+\mu)}{R} A_1 \frac{1}{\xi} \tag{3.26}$$

into the following equation, obtained from (1.11) and (3.16):

$$N_\rho = \frac{Eh}{1-\mu^2}\left[\frac{1}{L_0}\left(\frac{du}{d\xi} + \mu\frac{u}{\xi}\right) + (1+\mu)\frac{W}{R}\right]. \tag{3.27}$$

The first term of this equation, which only depends on u_0, then becomes:

$$N_\rho^0 = \frac{Eh}{R} A_1 \frac{1}{\xi^2}. \tag{3.28}$$

Comparing this with [the coefficient of $\frac{1}{\xi^2}$] in (3.13) we find that:

$$A_1 = C_8.$$

The analysis of a shallow spherical shell thus reduces to determining eight integration constants whose number corresponds to the order of the initial system of differential equations (1.1). The first four (C_1, C_2, C_3, C_4) determine the bending of the shell, while the last four (C_5, C_6, C_7, C_8) determine its deformation without bending.

The constant C_6 does not influence the states of stress and strain of the shell, and it can therefore be disregarded. The logarithmic terms in (3.13), (3.14), (3.24), must also vanish, since the logarithm is multivalued for doubly-connected regions which the shell may form, and does not therefore fulfill the requirement of uniqueness. Hence, the constant C_7 must be zero.

There remain thus six unknown constants. To determine them, three boundary conditions are required for each edge of the shell. Of each group of three, two will determine the bending deformation, and one the deformation without bending.

The boundary conditions corresponding to the bending deformation can be given in displacements W, W' (geometrical conditions), in forces and moments M_ρ, Q_ρ (statical conditions), or partly in forces and partly in displacements (mixed conditions).

The boundary conditions corresponding to the deformation without bending are determined either from the tangential displacements u or the value of the normal forces N_ρ.

§ 4. SHALLOW SPHERICAL SHELL SUBJECTED TO A UNIFORMLY DISTRIBUTED LOAD

1

Consider a shallow spherical shell on an elastic foundation, subjected to a uniformly distributed load p (Figure 126). Let the shell edges be free, so that the elastic foundation can be deformed beyond the limits of the structure. In accordance with (3.5), the normal displacements of the shell are:

$$W_1 = C_1 u_0(\xi) + C_2 v_0(\xi) + C_3 f_0(\xi) + C_4 g_0(\xi) + W_p, \tag{4.1}$$

where

$$W_p = \frac{pR^2}{Eh + kR^2}$$

is a particular integral of (3.2).

The following differential equation holds true for the region beyond the limits of the structure: [cf. (6.3) of Chapter IV]

$$\nabla_\xi^2 W_2 - \alpha_0^2 W_2 = 0, \tag{4.2}$$

The general integral of this equation can be represented in the form:

$$W_2 = B_1 I_0(\alpha_0\xi) + B_2 K_0(\alpha_0\xi), \tag{4.3}$$

where

$$\alpha_0^2 = \alpha^2 L_0^2 = \frac{kL_0^2}{2t}, \quad L_0 = \sqrt[4]{\frac{DR^2}{Eh + kR^2}}. \tag{4.4}$$

Here, B_1, and B_2 are integration constants, while $I_0(\alpha_0\xi)$ and $K_0(\alpha_0\xi)$ are modified zero-order Bessel and Hankel functions of the argument $\alpha_0\xi$.

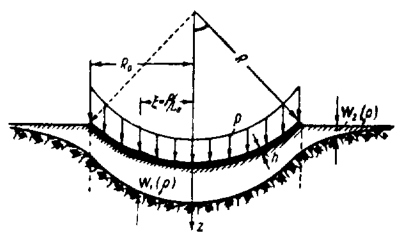


FIGURE 126.

The solution contains therefore six integration constants which are determined from the following boundary conditions (cf. the case of a circular plate, (6.8) through (6.10) of Chapter IV):

$$\left.\begin{array}{lll} \text{at } \rho = 0 \; (\xi = 0): & \text{a) } \frac{dW_1}{d\rho} = 0, & \text{b) } \int\limits_0 Q_\rho \rho \, d\theta = 0; \\ \text{at } \rho = R_0 \; \left(\xi = \frac{R_0}{L_0}\right): & \text{c) } M_\rho = 0, & \text{d) } Q_\rho = 2t\left(\frac{dW_2}{d\rho} - \frac{dW_1}{d\rho}\right)\cos\beta, \\ & \text{e) } W_1(R_0) = W_2(R_0); & \\ \text{at } \rho \to \infty \; (\xi \to \infty): & \text{f) } W_2(\rho) = 0. & \end{array}\right\} \tag{4.5}$$

The fourth boundary condition of (4.5) differs from the corresponding condition in (6.9) of Chapter IV by the coefficient $\cos\beta$, where β is half the central angle subtended by the shell. The reason for this is that the fictitious shearing force acting on the shell is equal to the projection of the fictitious shearing force acting on a circular plate, onto the normal to the middle surface of the shell at the edge.

Conditions a), b), and f) yield:

$$C_3 = C_4 = B_1 = 0, \tag{4.6}$$

so that the normal displacements of the shell are:

$$W_1 = C_1 u_0(\xi) + C_2 v_0(\xi) + \frac{pR^2}{Eh + kR^2}. \tag{4.7}$$

Condition e) yields:

$$B_2 = \frac{C_1 u_0(\xi_R) + C_2 v_0(\xi_R) + \dfrac{pR^2}{Eh + kR^2}}{K_0(\alpha_0 \xi_R)}, \tag{4.8}$$

where

$$\xi_R = \frac{R_0}{L_0}.$$

Substitution of expressions (2.7) for M_ρ and Q_ρ in conditions c) and d) yields:

$$\text{at} \quad \rho = R_0 \quad \left(\xi = \frac{R_0}{L_0}\right): \quad \left.\begin{aligned} -\frac{D}{L_0^2}\left[\nabla_\xi^2 W_1 - \frac{1-\mu}{\xi}\frac{dW_1}{d\xi}\right] &= 0, \\ \frac{d}{d\xi}\nabla_\xi^2 W_1 + \frac{2tL_0^2}{D}\left[\frac{dW_2}{d\xi} - \frac{dW_1}{d\xi}\right]\cos\beta &= 0. \end{aligned}\right\} \tag{4.9}$$

Substitution of (4.3), (4.6), (4.7), and (4.8) leads to the following two simultaneous equations in C_1 and C_2:

$$a_1 C_1 + a_2 C_2 = 0, \quad b_1 C_1 + b_2 C_2 = G_p, \tag{4.10}$$

where

$$\left.\begin{aligned} a_1 &= M_1(\xi_R) - (1-\mu)\overline{M}_1(\xi_R), \\ a_2 &= M_2(\xi_R) - (1-\mu)\overline{M}_2(\xi_R), \\ b_1 &= \left\{Q_1(\xi_R) + \frac{2tL_0^2}{D}\left[\theta_1(\xi_R)\cos\beta - u_0(\xi_R)\frac{K_1(\alpha_0\xi_R)\,\alpha_0\cos\beta}{K_0(\alpha_0\xi_R)}\right]\right\}, \\ b_2 &= \left\{Q_2(\xi_R) + \frac{2tL_0^2}{D}\left[\theta_2(\xi_R)\cos\beta - v_0(\xi_R)\frac{K_1(\alpha_0\xi_R)\,\alpha_0\cos\beta}{K_0(\alpha_0\xi_R)}\right]\right\}, \end{aligned}\right\} \tag{4.11}$$

$$G_p = \frac{pR^2}{Eh + kR^2}\frac{K_1(\alpha_0\xi_R)}{K_0(\alpha_0\xi_R)}. \tag{4.12}$$

The functions:

$$M_1, \quad M_2, \quad \overline{M}_1, \quad \overline{M}_2, \quad Q_1, \quad Q_2, \quad \theta_1, \quad \theta_2$$

appearing in (4.11) are given by (2.25) through (2.28) or series (6.20) of Chapter IV (see also (6.16) and (6.17) of Chapter IV.)

The solution of (4.10) is:

$$C_1 = -\frac{G_p a_2}{a_1 b_2 - a_2 b_1}, \quad C_2 = \frac{G_p a_1}{a_1 b_2 - a_2 b_1}. \tag{4.13}$$

The slopes, moments, and shearing forces of the shell are then by (4.7) [cf. (6.19) of Chapter IV]:

$$\left.\begin{aligned} W_1' &= -\frac{1}{L_0}[C_1\theta_1(\xi) + C_2\theta_2(\xi)], \\ M_\rho &= \frac{D}{L_0^2}[C_1 L_1(\xi) + C_2 L_2(\xi)], \\ M_\theta &= \frac{D}{L_0^2}[C_1 \overline{L}_1(\xi) + C_2 \overline{L}_2(\xi)], \\ Q_\rho &= -\frac{D}{L_0^3}[C_1 Q_1(\xi) + C_2 Q_2(\xi)]. \end{aligned}\right\} \tag{4.14}$$

2

Consider the state of plane stress of the shallow shell. By virtue of (4.7), the stress function Φ defined by (3.12) takes the form:

$$\Phi = \frac{EhL_0^2}{R}\Big[-C_1\varphi_1(\xi) - C_2\varphi_2(\xi) + C_5 + C_6\xi^2 + \\ + C_7\xi^2\ln\xi + C_8\ln\xi + \frac{pR^2}{Eh + kR^2}\frac{\xi^2}{4}\Big]. \tag{4.15}$$

The functions $\varphi_1(\xi), \varphi_2(\xi)$ are given here by (3.11) or by the corresponding series:

$$\left.\begin{aligned} \varphi_1(\xi) &= \sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{\xi}{2}\right)^{2m}\cos 2(m-1)\varphi, \\ \varphi_2(\xi) &= \sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{\xi}{2}\right)^{2m}\sin 2(m-1)\varphi. \end{aligned}\right\} \tag{4.16}$$

As already stated, the constants C_5 and C_7 in (4.15) vanish; we thus obtain:

$$\Phi = \frac{EhL_0^2}{R}\Big[-C_1\varphi_1(\xi) - C_2\varphi_2(\xi) + C_6\xi^2 + C_8\ln\xi + \frac{pR^2}{Eh + kR^2}\frac{\xi^2}{4}\Big]. \tag{4.17}$$

From (3.13) and (3.14) follows:

$$N_\rho = \frac{Eh}{R}\Big[C_1 n_1(\xi) + C_2 n_2(\xi) + 2C_6 + C_8\frac{1}{\xi^2} + \frac{1}{2}\frac{pR^2}{Eh + kR^2}\Big], \tag{4.18}$$

$$N_\theta = \frac{Eh}{R}\Big\{C_1[u_0(\xi) - n_1(\xi)] + C_2[v_0(\xi) - n_2(\xi)] + 2C_6 - C_8\frac{1}{\xi^2} + \frac{1}{2}\frac{pR^2}{Eh + kR^2}\Big\}, \tag{4.19}$$

where $n_1(\xi)$, and $n_2(\xi)$ are defined by (3.15), or by the corresponding series:

$$\left.\begin{aligned} n_1(\xi) &= \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,(m+1)!}\left(\frac{\xi}{2}\right)^{2m}\cos 2m\varphi,\\ n_2(\xi) &= \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,(m+1)!}\left(\frac{\xi}{2}\right)^{2m}\sin 2m\varphi.\end{aligned}\right\} \tag{4.20}$$

Hence, (3.24) becomes:

$$u = \frac{L_0}{R}\Big\{-(1+\mu)\Big[C_1\chi_1(\xi) + C_2\chi_2(\xi) + C_8\frac{1}{\xi} + \frac{1}{2}\frac{pR^2}{Eh+kR^2}\xi\Big] + 2C_6(1-\mu)\xi\Big\}, \tag{4.21}$$

where $\chi_1(\xi)$, $\chi_2(\xi)$ are defined by (3.25), or by the series:

$$\left.\begin{aligned} \chi_1(\xi) &= \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,(m+1)!}\left(\frac{\xi}{2}\right)^{2m+1}\cos 2m\varphi,\\ \chi_2(\xi) &= \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,(m+1)!}\left(\frac{\xi}{2}\right)^{2m+1}\sin 2m\varphi.\end{aligned}\right\} \tag{4.22}$$

The constants C_1 and C_2 in (4.17), (4.18), (4.19), and (4.21) are obtained from (4.14), while the constants C_6 and C_8 have to be determined from the boundary conditions for the tangential strains, which are:

$$\begin{aligned} &\text{at } \rho=0 \quad (\xi=0): && u=0;\\ &\text{at } \rho=R_0 \quad \left(\xi=\frac{R_0}{L_0}\right): && N_\rho = -Q^\Phi\sin\beta = -\frac{2l}{L_0}[W_2' - W_1']\sin\beta.\end{aligned} \tag{4.23}$$

The first condition states that the tangential displacements vanish at the shell apex; this is a consequence of the axial symmetry. By the second condition allowance is made for the existence along the shell contour of fictitious forces:

$$N^\Phi = -Q^\Phi\sin\beta, \tag{4.24}$$

equal in magnitude to the projection of the fictitious shearing forces acting on a circular plate, onto the tangent to the middle surface of the shell at the contour. The minus sign results from the convention according to which a negative value of N^Φ corresponds to a positive shearing force at the contour.

The first condition (4.23) yields:

$$C_8 = 0. \tag{4.25}$$

The second condition yields:

$$C_6 = \frac{1}{2}\left[\frac{RN^\Phi}{Eh} - C_1 n_1(\xi_R) - C_2 n_2(\xi_R) - \frac{1}{2}\frac{pR^2}{Eh+kR^2}\right], \tag{4.26}$$

where

$$N^{\Phi} = -\frac{2t}{L_0}\sin\beta \times$$

$$\times\left[C_1\theta_1(\xi_R) + C_2\theta_2(\xi_R) - \frac{C_1u_0(\xi_R) + C_2v_0(\xi_R) + \frac{pR^2}{Eh + kR^2}}{K_0(\alpha_0\xi_R)}\alpha_0 K_1(\alpha_0\xi_R)\right]. \qquad (4.27)$$

§ 5. SHALLOW SPHERICAL SHELL SUBJECTED TO A CONTOUR LOAD

1

Let a shell on an elastic foundation be subjected to a vertical contour load P_k [per unit length] (Figure 127).

The differential equation (3.2), is homogeneous in this case; its solution is:

$$W_1 = C_1u_0(\xi) + C_2v_0(\xi) + C_3f_0(\xi) + C_4g_0(\xi). \qquad (5.1)$$

For the region beyond the limits of the shell ($R_0 \leqslant \rho \leqslant \infty$) we have:

$$W_2 = B_1I_0(\alpha_0\xi) + B_2K_0(\alpha_0\xi). \qquad (5.2)$$

The constants in (5.1) and (5.2) are determined from the boundary conditions which can be formulated as in (4.5) with the exception of condition d), which is replaced by the following, having the same physical meaning:

$$\text{at } \rho = R_0 \quad \left(\xi = \frac{R_0}{L_0}\right) \qquad Q_\rho = \left[2t\left(\frac{dW_2}{d\rho} - \frac{dW_1}{d\rho}\right) + P_k\right]\cos\beta, \qquad (5.3)$$

where P_k is the vertical load per unit length of the contour.

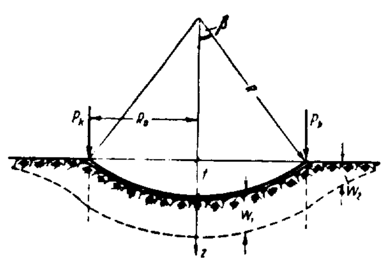


FIGURE 127.

Conditions a, b, e, and f of (4.5) yield:

$$\left.\begin{aligned} C_3 = C_4 = B_1 = 0, \\ B_2 = \frac{C_1u_0(\xi_R) + C_2v_0(\xi_R)}{K_0(\alpha_0\xi_R)}. \end{aligned}\right\} \qquad (5.4)$$

Using (4.9), (5.1), (5.2), and (5.4), the following two simultaneous equations in C_1 and C_2 are obtained from conditions c) and d):

$$\left.\begin{aligned} a_1C_1 + a_2C_2 &= 0, \\ b_1C_1 + b_2C_2 &= G_P. \end{aligned}\right\} \tag{5.5}$$

Their solution is:

$$C_1 = -\frac{G_P a_2}{a_1 b_2 - a_2 b_1}, \quad C_2 = \frac{G_P a_1}{a_1 b_2 - a_2 b_1}, \tag{5.6}$$

where a_1, a_2, b_1, b_2 are defined by (4.11), while:

$$G_P = -\frac{P_k L_0^3}{D} \cos\beta. \tag{5.7}$$

The normal displacements are thus:

$$W_1 = C_1 u_0(\xi) + C_2 v_0(\xi). \tag{5.8}$$

The slopes moments, and shearing forces are given by (4.14), the constants C_1 and C_2 being obtained from (5.6).

The normal forces are in the case considered [cf. (3.13) and (3.14)]:

$$\left.\begin{aligned} N_\rho &= \frac{Eh}{R}\left[C_1 n_1(\xi) + C_2 n_2(\xi) + 2C_6 + C_8 \frac{1}{\xi^2}\right], \\ N_\theta &= \frac{Eh}{R}\left\{C_1[u_0(\xi) - n_1(\xi)] + C_2[v_0(\xi) - n_2(\xi)] + 2C_6 - C_8 \frac{1}{\xi^2}\right\}, \end{aligned}\right\} \tag{5.9}$$

the tangential displacements being [cf. (3.24)]:

$$u = \frac{L_0}{R}\left\{-(1+\mu)\left[C_1\chi_1(\xi) + C_2\chi_2(\xi) + C_8\frac{1}{\xi}\right] + 2C_6(1-\mu)\xi\right\}. \tag{5.10}$$

The boundary conditions are:

$$u(0) = 0, \qquad N_\rho(R_0) = N^\Phi - P_k \sin\beta, \tag{5.11}$$

where N^Φ is the fictitious normal contour force given by (4.24). In the case considered [(cf. 4.27)]:

$$N^\Phi = -\frac{2t}{L_0}\left[C_1\theta_1(\xi_R) + C_2\theta_2(\xi_R) - \frac{C_1 u_0(\xi_R) + C_2 v_0(\xi_R)}{K_0(\alpha_0\xi_R)}\alpha_0 K_1(\alpha_0\xi_R)\right]\sin\beta. \tag{5.12}$$

The first and second boundary conditions (5.11) yield respectively:

$$C_8 = 0. \tag{5.13}$$

$$C_6 = \frac{1}{2}\left[\frac{(N^\Phi - P_k \sin\beta)R}{Eh} - C_1 n_1(\xi_R) - C_2 n_2(\xi_R)\right]. \tag{5.14}$$

2

If a horizontal load N_k is applied to [unit length of] the shell contour (Figure 128), the problem is solved in exactly the same way, the only difference being in the boundary condition (5.3), which become:

$$Q_\rho(R_0) = 2t\left(\frac{dW_2}{d\rho} - \frac{dW_1}{d\rho}\right)\cos\beta - N_k \sin\beta. \qquad (5.15)$$

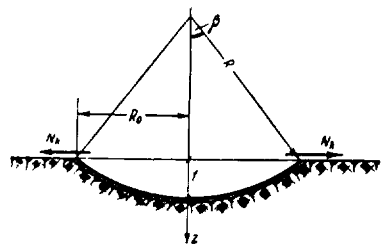


FIGURE 128.

The integration constants C_1 and C_2 are again determined by (5.6), the load term G_P being now:

$$G_N = \frac{N_k L_0^3}{D}\sin\beta. \qquad (5.16)$$

The forces are obtained from (5.9) and the displacements from (5.10). The corresponding boundary conditions are:

$$u(0) = 0, \quad N_\rho(R_0) = N^\Phi - N_k\cos\beta, \qquad (5.17)$$

which yield the following values for the integration constants C_6 and C_8 :

$$\left.\begin{aligned} C_8 &= 0, \\ C_6 &= \frac{1}{2}\left[\frac{(N^\Phi - N_k\cos\beta)R}{Eh} - C_1 n_1(\xi_R) - C_2 n_2(\xi_R)\right]. \end{aligned}\right\} \qquad (5.18)$$

Here N^Φ is the fictitious contour force, determined by (5.12).

3

Let the shell be subjected to moments M_k applied to [unit length of] its contour (Figure 129).

It is easily seen that the solution can in this case also be presented in the form of (5.1) and (5.2). The boundary conditions are given by (4.5), with the exception of condition c) which is to be replaced by:

$$M_\rho(R_0) = M_k. \qquad (5.19)$$

We obtain:

$$\left.\begin{aligned} &C_3 = C_4 = B_1 = 0, \\ &B_2 = \frac{C_1 u_0(\xi_R) + C_2 v_0(\xi_R)}{K_0(a_0 \xi_R)}. \end{aligned}\right\} \tag{5.20}$$

The following two simultaneous equations are obtained in C_1 and C_2:

$$\left.\begin{aligned} &a_1 C_1 + a_2 C_2 = G_M, \\ &b_1 C_1 + b_2 C_2 = 0, \end{aligned}\right\} \tag{5.21}$$

The magnitudes a_1, a_2, b_1, and b_2 are defined by (4.11), while:

$$G_M = \frac{M_k L_0^2}{D}. \tag{5.22}$$

The solution of (5.21) is:

$$C_1 = \frac{G_M b_2}{a_1 b_2 - a_2 b_1}, \quad C_2 = -\frac{G_M b_1}{a_1 b_2 - a_2 b_1}. \tag{5.23}$$

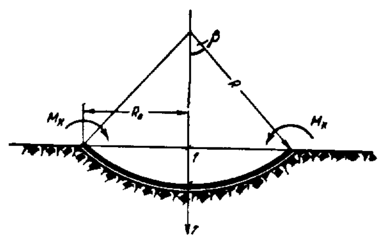


FIGURE 129.

The forces and displacements are in this case again given by (5.9) and (5.10), with $C_8 = 0$ and:

$$C_6 = \frac{1}{2}\left[\frac{RN^{\Phi}}{Eh} - C_1 n_1(\xi_R) - C_2 n_2(\xi_R)\right]. \tag{5.24}$$

§ 6. APPROXIMATIVE ANALYSIS OF A SHALLOW SPHERICAL SHELL ON AN ELASTIC FOUNDATION

1

Taking into account its small deformability in relation to that of the elastic foundation, we shall consider the shallow shell as a rigid punch.

The reactions of the elastic foundation are then determined by (3.10) and (3.12) of Chapter IV, valid for a circular punch of radius R_0 (Figure 130, a):

$$q = \frac{P_0}{\pi R_0^2\left[1 + 2\frac{K_1(\alpha R_0)}{K_0(\alpha R_0)\,\alpha R_0}\right]}, \tag{6.1}$$

$$Q^{\Phi} = \frac{P_0}{\pi R_0\left[1 + 2\frac{K_1(\alpha R_0)}{K_0(\alpha R_0)\,\alpha R_0}\right]}\,\frac{K_1(\alpha R_0)}{K_0(\alpha R_0)\,\alpha R_0}, \tag{6.2}$$

where P_0 is the resultant of the given vertical load.

The analysis of a shell on an elastic foundation thus reduces to determining the strains and stresses in the shell subjected to a given external load and to the reactions q and Q^{Φ} of the elastic foundation, all these forces being in equilibrium. For a uniformly distributed external load p, the system is shown in Figure 130, b, where:

$$p^* = p - q, \tag{6.3}$$

and

$$Q^{\Phi} = \frac{\pi R_0^2 p^*}{2\pi R_0} = \frac{p^* R_0}{2}. \tag{6.4}$$

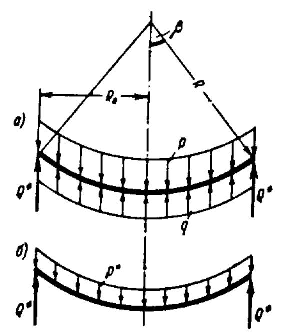


FIGURE 130.

2

It was shown in section 1 that the problem of a shallow spherical shell subjected to a vertical distributed load p^* reduces to integrating the two differential equations [cf. (1.1), (1.7)]:

$$\left.\begin{aligned} D\nabla^2\nabla^2 W + \frac{Eh}{R^2}W - p^* = 0,\\ \frac{R}{Eh}\nabla^2\nabla^2\Phi - \nabla^2 W = 0, \end{aligned}\right\} \tag{6.5}$$

the first of which yields the displacement function W, and the second the stress function Φ.

We introduce the dimensionless coordinates:

$$\xi = \frac{\rho}{L_0},$$

where

$$L_0 = \sqrt[4]{\frac{DR^2}{Eh}}, \qquad (6.6)$$

We can then rewrite (6.5) as follows:

$$\nabla_\xi^2 \nabla_\xi^2 W + W = \frac{p^* L_0^4}{D}, \qquad (6.7)$$

$$\nabla_\xi^2 \nabla_\xi^2 \Phi = \frac{EhL_0^2}{R} \nabla_\xi^2 W, \qquad (6.8)$$

where

$$\nabla_\xi^2 = \frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi}. \qquad (6.9)$$

3

The homogeneous equation obtained from (6.7) for $p^* = 0$ can be reduced to an equivalent system of two second-order equations:

$$\left.\begin{aligned} \frac{d^2W}{d\xi^2} + \frac{1}{\xi}\frac{dW}{d\xi} + iW = 0, \\ \frac{d^2W}{d\xi^2} + \frac{1}{\xi}\frac{dW}{d\xi} - iW = 0. \end{aligned}\right\} \qquad (6.10)$$

The general integral of this system is:

$$W = A_1 J_0(\xi\sqrt{i}) + A_2 J_0(\xi\sqrt{-i}) + A_3 H_0^{(1)}(\xi\sqrt{i}) + A_4 H_0^{(2)}(\xi\sqrt{-i}), \qquad (6.11)$$

where $J_0(\xi\sqrt{i})$ and $J_0(\xi\sqrt{-i})$ are zero-order Bessel functions of the first kind, of the arguments $(\xi\sqrt{i})$ and $(\xi\sqrt{-i})$; $H_0^{(1)}(\xi\sqrt{i})$ and $H_0^{(2)}(\xi\sqrt{-i})$ are respectively zero-order Hankel functions of the first and second kind, of the same arguments.

The only difference between (6.11) and (2.19) of Chapter IV is in the arguments of the functions considered. Integral (6.11) is a particular case of the more general solution (2.19) of Chapter IV, since the straight lines along which the Bessel and Hankel functions in (6.11) are determined make angles of 45° with the real axis in the complex plane:

$$\left.\begin{aligned} \varphi = \arg\sqrt{i} = 45°, \\ \varphi = \arg\sqrt{-i} = -45°. \end{aligned}\right\} \qquad (6.12)$$

It is convenient to express (6.11) through real functions:

$$W = C_1 u_0(\xi) + C_2 v_0(\xi) + C_3 f_0(\xi) + C_4 g_0(\xi), \tag{6.13}$$

where

$$\left.\begin{aligned} u_0(\xi) &= \operatorname{Re} J_0(\sqrt{i}\,\xi), & f_0(\xi) &= \operatorname{Re} H_0^{(1)}(\sqrt{i}\,\xi), \\ v_0(\xi) &= \operatorname{Im} J_0(\sqrt{i}\,\xi), & g_0(\xi) &= \operatorname{Im} H_0^{(2)}(\sqrt{-i}\,\xi). \end{aligned}\right\} \tag{6.14}$$ *

The following relationships are obtained from (2.20) of Chapter IV and (6.12) of this chapter :

$$\nabla^2 u_0 = v_0, \quad \nabla^2 v_0 = -u_0, \quad \nabla^2 f_0 = g_0, \quad \nabla^2 g_0 = -f_0. \tag{6.15}$$

The general integral of the nonhomogeneous equation (6.7) is then:

$$W = C_1 u_0(\xi) + C_2 v_0(\xi) + C_3 f_0(\xi) + C_4 g_0(\xi) + W_p, \tag{6.16}$$

where W_p is a particular integral of (6.7).

For instance, if $p^* = \text{const}$ we can put:

$$W_p = \frac{p^* R^2}{Eh} \tag{6.17}$$

From the boundary conditions a) and b) of (4.5) we obtain:

$$C_3 = C_4 = 0. \tag{6.18}$$

Substitution of (6.17) and (6.18) in (6.16) yields:

$$W = C_1 u_0(\xi) + C_2 v_0(\xi) + \frac{p^* R^2}{Eh}. \tag{6.19}$$

We then obtain for $p^* = \text{const}$ from (2.7), using (6.15) and (6.19):

$$\left.\begin{aligned} \frac{dW}{d\rho} &= \frac{1}{L_0}[C_1 u_0'(\xi) + C_2 v_0'(\xi)], \\ M_\rho &= -\frac{D}{L_0^2}\left\{C_1\left[v_0(\xi) - (1-\mu)\frac{u_0'(\xi)}{\xi}\right] + \right. \\ &\qquad \left. + C_2\left[-u_0(\xi) - (1-\mu)\frac{v_0'(\xi)}{\xi}\right]\right\}, \\ M_\theta &= -\frac{D}{L_0^2}\left\{C_1\left[\mu v_0(\xi) + (1-\mu)\frac{u_0'(\xi)}{\xi}\right] + \right. \\ &\qquad \left. + C_2\left[-\mu u_0(\xi) + (1-\mu)\frac{v_0(\xi)}{\xi}\right]\right\}, \\ Q_\rho &= -\frac{D}{L_0^3}[C_1 v_0'(\xi) - C_2 u_0'(\xi)]. \end{aligned}\right\} \tag{6.20}$$

* These functions are known as Thomson functions. They, and their derivatives, are tabulated in the appendix (Table 11).

These expressions could also have been obtained from (4.14) by substituting in them (2.25) through (2.28) of Chapter IV, taking into account (6.12).

Equations (4.17), (4.18), (4.19), (4.21) are again valid. Taking into account (4.25), they become in the case considered:

$$\Phi = \frac{EhL_0^2}{P}\left[-C_1 v_0(\xi) + C_2 u_0(\xi) + C_6\xi^2 + \frac{p^* R^2}{Eh}\frac{\xi^2}{4}\right], \tag{6.21}$$

$$N_\rho = \frac{Eh}{R}\left[-C_1\frac{v_0'(\xi)}{\xi} + C_2\frac{u_0'(\xi)}{\xi} + 2C_6 + \frac{1}{2}\frac{p^* R^2}{Eh}\right], \tag{6.22'}$$

$$N_\theta = \frac{Eh}{R}\left\{C_1\left[u_0(\xi) + \frac{v_0'(\xi)}{\xi}\right] + C_2\left[v_0(\xi) - \frac{u_0'(\xi)}{\xi}\right] + 2C_6 + \frac{1}{2}\frac{p^* R^2}{Eh}\right\}, \tag{6.22''}$$

$$u = \frac{L_0}{R}\left\{-(1+\mu)\left[-C_1 v_0'(\xi) + C_2 u_0'(\xi)\right] + {} + 2C_6(1-\mu)\xi - \frac{(1+\mu)}{2}\frac{p^* R^2}{Eh}\xi\right\}. \tag{6.23}$$

These expressions contain three integration constants which are determined from the boundary conditions at the shell contour.

5

Consider as numerical example the shallow spherical shell shown in Figure 131.

Let the geometrical and physical characteristics of the shell and the elastic foundation be as follows*:

$$\left.\begin{aligned}
R &= 13.5 \text{ m},\\
h &= 0.46 \text{ m},\\
R_0 &= 5.1 \text{ m},\\
E_{IS} &= 4\cdot 10^3 \text{ cm/m}^2,\\
\nu_s &= 0.4,\\
H/R_0 &= 1.0,\\
f &= 1.0 \text{ m}.\\
E &= 2\cdot 10^6 \text{ cm/m}^2,\\
\mu &= 0.167,\\
\gamma &= 1.55,\\
P &= 1.0 \text{ cm/m},\\
D &= 16.2\cdot 10^4 \text{ cm/m}.
\end{aligned}\right\} \tag{6.24}$$

We then obtain from (3.16) and (3.17) of Chapter IV:

$$\left.\begin{aligned}
k &= 1.6\cdot 10^3 \text{ m/m}^3 \quad & 2t &= 1.85\cdot 10^3 \text{ m/m},\\
\alpha &= 0.93 \text{ 1/m}, & \alpha R_0 &= 4.72.
\end{aligned}\right\} \tag{6.25}$$

By (6.1) and (6.2), the reactions of the elastic foundation are:

$$q = 0.269 \text{ m/m}^2 \qquad Q^\Phi = 0.315 \text{ m/m}, \tag{6.26}$$

* It is assumed that the function $\psi(z)$ determining the vertical distribution of the displacements in the elastic foundation is given by (3.15) of Chapter IV.

Their substitution in (6.3) yields:

$$p^* = -0.269 \text{ m/m}^2. \tag{6.27}$$

Insertion of the values (6.24) into (6.6) gives:

$$L_0 = 1.34 \text{ m}, \quad \xi_R = \frac{R_0}{L_0} = 3{,}8. \tag{6.28}$$

The functions entering in (6.19) through (6.23) are for $\xi_R = 3.8$:

$$\left.\begin{aligned} u_0 &= -1.967, & u_0' &= -2.822, \\ v_0 &= -2.345, & v_0' &= 0.0526 \\ \frac{u_0'}{\xi_R} &= -0.742, & \frac{v_0'}{\xi_R} &= 0.138. \end{aligned}\right\} \tag{6.29}$$

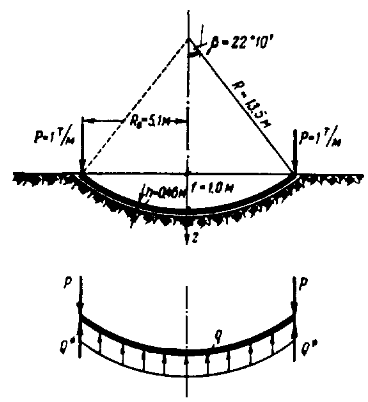


FIGURE 131.

The boundary conditions are in the case considered:

at $\rho = R_0\left(\xi_R = \frac{R_0}{L_0}\right)$: $\qquad M_\rho = 0,$

$$\left.\begin{aligned} Q_\rho &= (P - Q^\Phi)\cos\beta, \\ N_\rho &= -(P - Q^\Phi)\sin\beta, \end{aligned}\right\} \tag{6.30}$$

or [by (6.20) and (6.22')):

$$\left.\begin{aligned} &C_1\left[v_0(\xi_R) - (1-\mu)\frac{u_0'(\xi_R)}{\xi_R}\right] + \\ &\qquad + C_2\left[-u_0(\xi_R) - (1-\mu)\frac{v_0'(\xi_R)}{\xi_R}\right] = 0, \\ &C_1 v_0'(\xi_R) - C_2 u_0'(\xi_R) = -\frac{L_0^3(P - Q^\Phi)\cos\beta}{D}, \\ &-C_1\frac{v_0'(\xi_R)}{\xi_R} + C_2\frac{u_0'(\xi_R)}{\xi_R} + 2C_4 + \frac{1}{2}\frac{p^* R^2}{Eh} = \\ &\qquad = -\frac{R(P - Q^\Phi)}{Eh}\sin\beta. \end{aligned}\right\} \tag{6.31}$$

Substitution of (6.26), (6.28), (6.29) in (6.31) yields finally:

$$C_1 = -3.68 \cdot 10^{-6} \text{ m}, \; C_2 = -3.26 \cdot 10^{-6} \text{ m}, \; C_6 = -0.948 \cdot 10^{-6} \text{m}, \qquad (6.32)$$

The moments M_ρ and M_θ, and the forces N_ρ and N_θ, obtained from (6.20) and (6.22) for the numerical values (6.32), have been plotted in Figures 132 through 135.

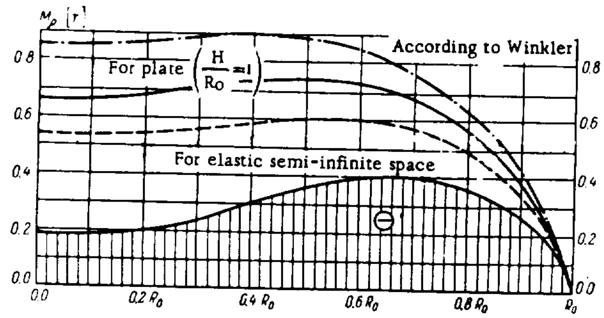


FIGURE 132.

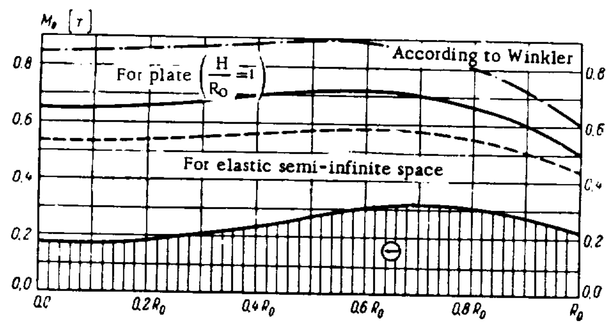


FIGURE 133.

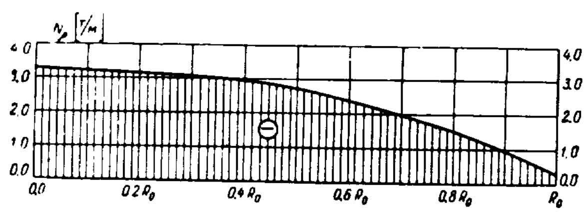


FIGURE 134.

Figures 132 and 133 also show values of the moments M_ρ and M_θ, calculated for a circular plate of radius $R = R_0$, with the same values of the characteristics of the elastic foundation and of the structure. The calculations were performed by three different methods: the method employed in Chapter IV (for $H/R_0 = 1$, $\gamma = 1.55$), the method of the elastic semi-infinite space /26/, and by Winkler's method, the foundation modulus being:

$$k = 1.6 \cdot 10^3 \text{ m/m}^3.$$

Comparison of these curves shows that the bending moments M_ρ, and M_0 acting on a shallow spherical shell are considerably smaller than those acting on a circular plate.

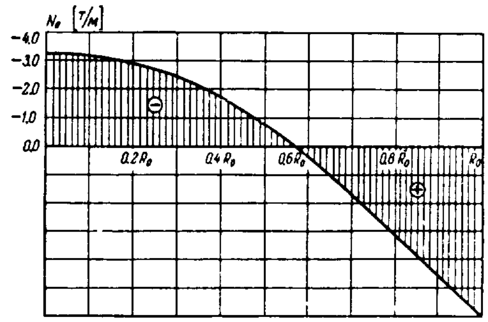


FIGURE 135.

§ 7. APPLICATION OF THE ABOVE METHOD TO THE ANALYSIS OF THE BOTTOMS OF CYLINDRICAL RESERVOIRS

The preceding sections dealt with some problems concerning a shallow spherical shell on an elastic foundation. It was assumed that the shell edges are free and that the shell is acted upon either by a uniformly distributed or by a contour load (Figures 126 through 129).

We shall now consider the case when the spherical shell forms the bottom of a cylindrical reservoir on an elastic foundation (Figure 136).

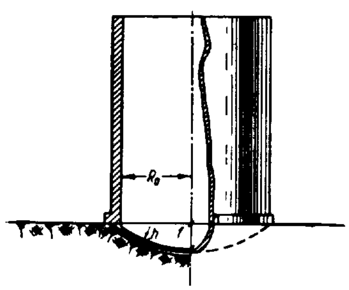


FIGURE 136.

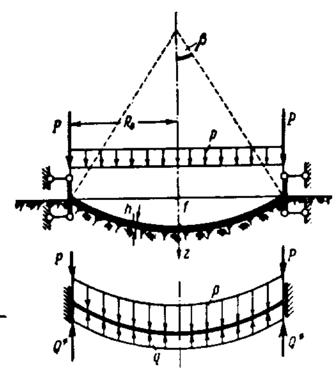


FIGURE 137.

1

The bottom is usually secured to the cylinder by a supporting ring and can be assumed in a first approximation to be a spherical shell built-in along its contour. In other words, we consider the system shown schematically in Figure 137, disregarding the remainder of the structure. In this figure, the contour forces P represent the load transmitted to the shell by the reservoir walls, while the uniformly distributed load p represents the pressure of the liquid in the reservoir. The reactions q and Q^{Φ} of the elastic foundation are obtained from (6.1) and (6.2), assuming that in relation to the elastic foundation the shell can be considered as a rigid punch.

As before, the stresses and strains of the shell are determined by (6.19) through (6.23). These contain the three integration constants C_1, C_2, and C_6, which can be determined from the following boundary conditions:

$$\text{at } \rho = R_0 \left(\xi = \frac{R_0}{L_0}\right) \qquad W = 0, \quad W' = 0, \quad u = 0. \tag{7.1}$$

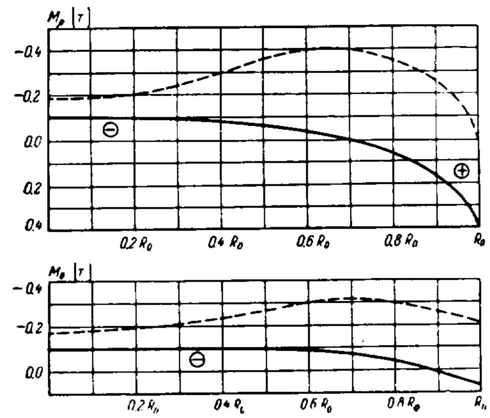


FIGURE 138.

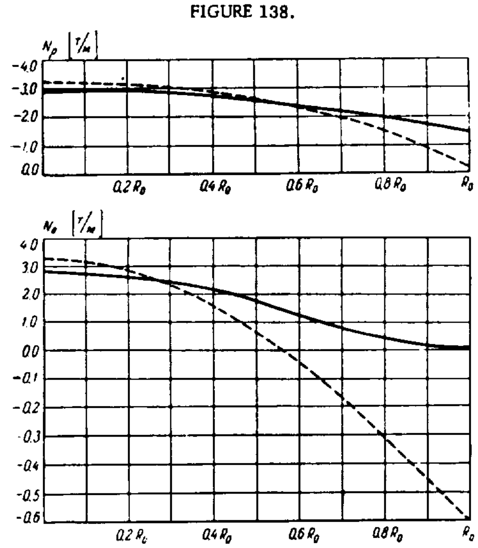


FIGURE 139.

These conditions yield:

$$\left.\begin{aligned}&C_1u_0(\xi_R)+C_2v_0(\xi_R)=-\frac{p^*R^2}{Eh},\\&C_1u_0'(\xi_R)+C_2v_0'(\xi_R)=0,\\&-(1+\mu)[-C_1v_0'(\xi_R)+C_2u_0'(\xi_R)]+2C_6(1-\mu)\xi_R-\\&\qquad-\frac{(1+\mu)}{2}\frac{p^*R^2}{Eh}\xi_R=0,\end{aligned}\right\}\quad(7.2)$$

where $p^*=p-q$.

The solution of these equations is:

$$\left.\begin{aligned}&C_1=-\frac{v_0'(\xi_R)}{[u_0(\xi_R)v_0'(\xi_R)-u_0'(\xi_R)v_0(\xi_R)]}\frac{p^*R^2}{Eh},\\&C_2=\frac{u_0'(\xi_R)}{[u_0(\xi_R)v_0'(\xi_R)-u_0'(\xi_R)v_0(\xi_R)]}\frac{p^*R^2}{Eh},\\&C_6=\frac{(1+\mu)}{4(1-\mu)}\left\{\frac{2[u_0'^2(\xi_R)+v_0'^2(\xi_R)]}{\xi_R[u_0(\xi_R)v_0'(\xi_R)-u_0'(\xi_R)v_0(\xi_R)]}+1\right\}\frac{p^*R^2}{Eh}.\end{aligned}\right\}\quad(7.3)$$

The bending moments M_ρ and M_θ, and the forces N_ρ and N_θ, calculated by means of (7.3), (6.20), and (6.22) for a reservoir bottom subjected only to a contour load P, have been plotted in Figures 138 and 139. The values of the elastic constants of the soil and the structure are given by (6.24). The results obtained for a shell with free edges have been plotted in broken line in the same figures.

It is seen that reinforcing the shell along its contour by a rigid ring preventing radial displacement and rotation, reduces to a certain degree the values of M_ρ, M_θ, and N_ρ, while the tension N_θ at the shell contour becomes considerably less. This is very important when a shell forms the bottom of a reservoir.

2

If the supporting ring prevents only radial displacement, the shell can be analyzed according to the scheme shown in Figure 140.

The boundary conditions are in this case:

$$\text{at } \rho=R_0\quad\left(\xi=\frac{R_0}{L_0}\right):\qquad W=0,\quad M_\rho=0,\quad u=0.\qquad(7.4)$$

or

$$\left.\begin{aligned}&C_1u_0(\xi_R)+C_2v_0(\xi_R)=-\frac{p^*R^2}{Eh},\\&C_1\left[v_0(\xi_R)-(1-\mu)\frac{u_0'(\xi_R)}{\xi_R}\right]+\\&\qquad+C_2\left[-u_0(\xi_R)-(1-\mu)\frac{v_0'(\xi_R)}{\xi_R}\right]=0,\\&-(1+\mu)[-C_1v_0'(\xi_R)+C_2u_0'(\xi_R)]+\\&\qquad+2C_6(1-\mu)\xi_R-\frac{(1+\mu)}{2}\frac{p^*R^2}{Eh}\xi_R=0.\end{aligned}\right\}\quad(7.5)$$

The integration constants are thus:

$$
\left.
\begin{aligned}
C_1 &= \frac{K_1}{[u_0(\xi_R)K_1+v_0(\xi_R)K_2]}\,\frac{p^*R^2}{Eh},\\
C_2 &= \frac{K_2}{[u_0(\xi_R)K_1+v_0(\xi_R)K_2]}\,\frac{p^*R^2}{Eh},\\
C_6 &= \frac{(1+\mu)}{4(1-\mu)}\left[\frac{-v_0'(\xi_R)K_1-u_0'(\xi_R)K_2}{\xi_R[u_0(\xi_R)K_1+v_0(\xi_R)K_2]}+1\right]\frac{p^*R^2}{Eh},\\
\text{where}\qquad K_1 &= u_0(\xi_R)+(1-\mu)\frac{v_0'(\xi_R)}{\xi_R},\\
K_2 &= v_0(\xi_R)+(1-\mu)\frac{u_0'(\xi_R)}{\xi_R}.
\end{aligned}
\right\}
\qquad (7.6)
$$

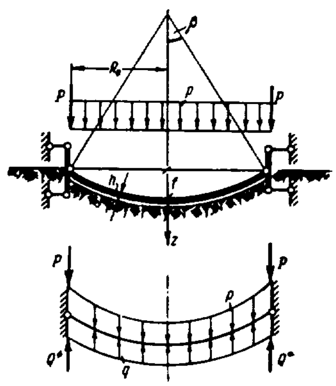


FIGURE 140.

3.

To obtain a more accurate solution it is necessary to make allowance for the effect of the cylindrical reservoir walls on the strains of the bottom. This can be done by the methods used to analyze statically indeterminate systems, i.e., the method of forces or the method of displacements (strains).

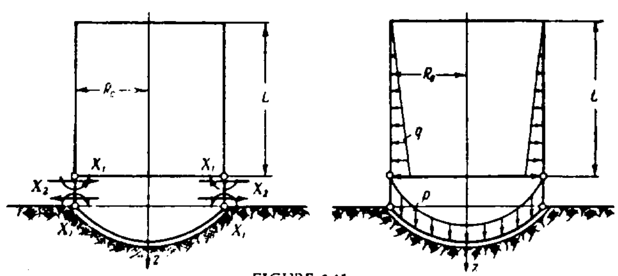


FIGURE 141.

In the first method, a cut is made in the zone where the bottom joins the cylindrical reservoir wall, and the constraints there are replaced by unknown forces [and moments] (Figure 141). In accordance with the convention adopted for the signs, the canonical equations expressing the continuity of the deformations are:

$$\left.\begin{aligned}(\Delta_{11}^{c}+\Delta_{11}^{b})X_1+(\Delta_{12}^{c}-\Delta_{12}^{b})X_2-\Delta_{1q}^{c}+\Delta_{1p}^{b}-\Delta_{1P}^{b}=0,\\(\Delta_{21}^{c}+\Delta_{21}^{b})X_1+(\Delta_{22}^{c}+\Delta_{22}^{b})X_2-\Delta_{2q}^{c}+\Delta_{2p}^{b}+\Delta_{2P}^{b}=0.\end{aligned}\right\}\qquad(7.7)$$

The first equation states that there is no relative rotation at the cut; the second states that there is no relative displacement in the direction of X_2.

The coefficients and load terms in (7.7) are given in absolute values. The first subscript indicates the place and direction of the displacement. the second subscript, its cause. The superscript "c" refers to the cylindrical reservoir wall, the superscript "b" to the bottom.

The load terms:

$$\Delta_{1q}^{c},\ \Delta_{2q}^{c},\ \Delta_{1p}^{b},\ \Delta_{2p}^{b}$$

define the absolute values of the displacements due to the external loads q and p, while the load terms Δ_{1P}^{b}, Δ_{2P}^{b} correspond to the displacements due to the contour load P transmitted by the reservoir wall to the bottom.

All coefficients and load terms with superscript "b" can be obtained from (6.19), (6.20), and (6.23).

To obtain the coefficients with superscript "c", it is necessary to consider the axisymmetrical deformation of a cylindrical shell subjected both to an external radial load q and to contour forces X_2 and moments X_1. This problem reduces to solving a fourth-order differential equation identical in form with the differential equation of the bending of a beam on an elastic Winkler foundation.

This problem is discussed in detail in many books* so that the coefficients with superscript "c" can be found without difficulty.

The same procedure is adopted when the reservoir bottom is a circular plate instead of a spherical shell. The coefficients with superscript "b" are in this case obtained from the formulas of sections 3 and 6 of Chapter IV.

* See for instance /8/.

Chapter VI

DYNAMICS AND STABILITY OF BEAMS AND PLATES ON ELASTIC FOUNDATIONS

§ 1. DIFFERENTIAL EQUATION OF THE VIBRATIONS OF A BEAM ON AN ELASTIC SINGLE-LAYER FOUNDATION

Consider a beam of width δ on an elastic foundation, subjected to an external load $p(x, t)$ varying with time (Figure 142). This is the case of dynamic loading, where the inertia forces acting in the deformed system become significant.

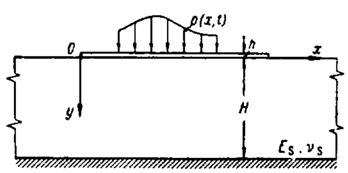


FIGURE 142.

The differential equation of motion of the beam is:

$$EJV^{IV}(x, t) = p(x, t) - m_1 \frac{\partial^2 V(x, t)}{\partial t^2} - q(x, t), \tag{1.1}$$

where EJ = rigidity of beam, $J = \frac{\delta h^3}{12(1-\mu^2)}$,

m_1 = mass of beam per unit length,

$q(x, t)$ = distributed reaction of foundation, due to elasticity and inertia of soil.

To determine the reactions $q(x, t)$, we cut out an elementary column from the elastic foundation (cf. sections 2 and 3 of Chapter I), and consider the equilibrium conditions of this column, applying Lagrange's principle of virtual displacements. Assuming that no horizontal displacements occur in the single-layer foundation and that the vertical displacements are described by the function $\psi(y)$, the virtual work done by all external and internal forces acting on this column becomes:

$$\frac{E_0 \delta}{2(1+\nu_0)} V'' \int_0^H \psi^2(y)\, dy - \frac{E_0 \delta}{1-\nu_0^2} V \int_0^H \psi'^2(y) dy - \overline{m}_0 \delta \int_0^H \psi^2(y)\, dy \frac{\partial^2 V}{\partial t^2} + q(x, t) = 0, \tag{1.2}$$

where $q(x,t)$ = load per unit length, applied to surface of elastic foundation (foundation reaction); $\bar{m}_0 = \frac{\gamma_0}{g}$ = mass per unit volume of foundation (γ_0 = specific weight of soil, g = gravitational acceleration); also,

$$E_0 = \frac{E_s}{1-\nu_s^2}, \quad \nu_0 = \frac{\nu_s}{1-\nu_s}, \tag{1.3}$$

where E_s and ν_s = modulus of elasticity and Poisson's ratio of elastic foundation respectively.

We introduce the symbols:

$$\left.\begin{aligned} k &= \frac{E_0\delta}{1-\nu_0^2}\int_0^H \psi'^2(y)\,dy, \\ t &= \frac{E_0\delta}{4(1+\nu_0)}\int_0^H \psi^2(y)\,dy, \\ m_0 &= \bar{m}\delta\int_0^H \psi^2(y)\,dy, \end{aligned}\right\} \tag{1.4}$$

Equation (1.2) then becomes:

$$2tV'' - kV - m_0\frac{\partial^2 V}{\partial t^2} + q(x,t) = 0. \tag{1.5}$$

This is the differential equation of the vibrations of a single-layer foundation under the action of a load $q(x,t)$. Eliminating $q(x,t)$ between (1.1) and (1.5), we obtain:

$$V^{IV} - 2r^2V'' + s^4V + m^*\frac{\partial^2 V}{\partial t^2} = \frac{p(x,t)}{EJ}, \tag{1.6}$$

where

$$\left.\begin{aligned} s^4 &= \frac{k}{EJ} = \frac{E_0\delta}{EJ(1-\nu_0^2)}\int_0^H \psi'^2(y)\,dy, \\ r^2 &= \frac{t}{EJ} = \frac{E_0\delta}{4EJ(1+\nu_0)}\int_0^H \psi^2(y)\,dy, \\ m^* &= \frac{m_1+m_0}{EJ} = \left(\frac{\gamma\delta}{g}h + \frac{\gamma_0\delta}{g}\int_0^H \psi^2(y)\,dy\right)\frac{1}{EJ} \end{aligned}\right\} \tag{1.7}$$

(γ and γ_0 = specific weights of materials of beam and elastic foundation respectively).

The partial differential equation (1.6) thus shows that not only the beam, but also the elastic foundation vibrates.

§ 2. FREE VIBRATIONS OF A BEAM

1

When $p(x, t) = 0$, (1.6) reduces to:

$$V^{IV} - 2r^2V'' + s^4V = -m^*\frac{\partial^2 V}{dt^2}. \tag{2.1}$$

This equation describes the free vibrations of the beam when it is disturbed from its state of equilibrium and then left to itself.

We assume a solution in the form of a product of two functions, one of which depends only on x, the other only on t:

$$V = X(x)\,T(t). \tag{2.2}$$

Substitution of (2.2) in (2.1) yields:

$$\frac{X^{IV} - 2r^2X'' + s^4X}{X} = -\frac{m^*T''}{T}. \tag{2.3}$$

Since its left-hand side is a function of x only, and its right-hand side of t only, (2.3) will hold in the general case only if each side is equal to the same constant. Denoting this constant by $m^*\omega^2$, we obtain:

$$T'' + \omega^2T = 0, \tag{2.4}$$

$$X^{IV} - 2r^2X'' + (s^4 - m^*\omega^2)\,X = 0. \tag{2.5}$$

The solution of (2.1) thus reduces to integrating two ordinary differential equations.

Equation (2.4) has the same form as the equation of free vibrations of a system with one degree of freedom: it describes a simple harmonic motion with frequency ω. The solution of this equation is:

$$T = A\sin\omega t + B\cos\omega t, \tag{2.6}$$

where A and B = constants determined from initial conditions.

The mode of the free vibrations of the beam is determined by (2.5), which can also be written thus:

$$X^{IV} - 2r^2X'' - (\lambda^4 - r^4)\,X = 0, \tag{2.7}$$

where

$$\lambda^4 = r^4 - s^4 + m^*\omega^2. \tag{2.8}$$

The general integral of (2.7) is:

$$X(x) = C_1\,\mathrm{sh}\,\alpha x + C_2\,\mathrm{ch}\,\alpha x + C_3\sin\beta x + C_4\cos\beta x,$$

where $C_1, \ldots, C_4$ are constants, and:

$$\alpha^2 = \lambda^2 + r^2, \quad \beta^2 = \lambda^2 - r^2. \tag{2.9}$$

Since ω is still unknown, the parameter λ^2, on which α and β depend, will also be unknown.

The general solution of (2.7) contains therefore the five unknowns C_1, C_2, C_3, C_4, and λ, which can be determined from the statical, kinematic, or mixed-type boundary conditions at the beam ends. Only two conditions can be formulated for each end. When no external forces act, the boundary conditions are homogeneous, containing only the values of the function $X(x)$ or of its derivatives at $x=0$ and $x=l$.

By expanding the boundary conditions at the beam ends $x=0$ and $x=l$ with the aid of the formulas:

$$\left.\begin{aligned} X(x) &= C_1 \operatorname{sh} \alpha x + C_2 \operatorname{ch} \alpha x + C_3 \sin \beta x + C_4 \cos \beta x, \\ X'(x) &= C_1 \alpha \operatorname{ch} \alpha x + C_2 \alpha \operatorname{sh} \alpha x + C_3 \beta \cos \beta x - C_4 \beta \sin \beta x, \\ X''(x) &= C_1 \alpha^2 \operatorname{sh} \alpha x + C_2 \alpha^2 \operatorname{ch} \alpha x - C_3 \beta^2 \sin \beta x - C_4 \beta^2 \cos \beta x, \\ X'''(x) &= C_1 \alpha^3 \operatorname{ch} \alpha x + C_2 \alpha^3 \operatorname{sh} \alpha x - C_3 \beta^3 \cos \beta x + C_4 \beta^3 \sin \beta x, \end{aligned}\right\} \tag{2.10}$$

we obtain, putting $x=0$ and $x=l$, a system of four linear equations in the four integration constants $C_1, \ldots, C_4$. Since the boundary conditions are homogeneous, the four equations formed will also be homogeneous. In the general case we obtain:

$$\left.\begin{aligned} a_{11}(\lambda) C_1 + a_{12}(\lambda) C_2 + a_{13}(\lambda) C_3 + a_{14}(\lambda) C_4 = 0, \\ a_{21}(\lambda) C_1 + a_{22}(\lambda) C_2 + a_{23}(\lambda) C_3 + a_{24}(\lambda) C_4 = 0, \\ a_{31}(\lambda) C_1 + a_{32}(\lambda) C_2 + a_{33}(\lambda) C_3 + a_{34}(\lambda) C_4 = 0, \\ a_{41}(\lambda) C_1 + a_{42}(\lambda) C_2 + a_{43}(\lambda) C_3 + a_{44}(\lambda) C_4 = 0, \end{aligned}\right\} \tag{2.11}$$

where the coefficients $a_{ik}(\lambda)$ $(i, k = 1, 2, 3, 4)$ are functions of λ. By equating to zero the determinant of system (2.11), we obtain an equation in λ:

$$\Delta(\lambda) = \begin{vmatrix} a_{11}(\lambda) & a_{12}(\lambda) & a_{13}(\lambda) & a_{14}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) & a_{23}(\lambda) & a_{24}(\lambda) \\ a_{31}(\lambda) & a_{32}(\lambda) & a_{33}(\lambda) & a_{34}(\lambda) \\ a_{41}(\lambda) & a_{42}(\lambda) & a_{43}(\lambda) & a_{44}(\lambda). \end{vmatrix} = 0. \tag{2.12}$$

This equation is called characteristic equation of the homogeneous boundary-value problem, i.e., the problem described by the homogeneous differential equation (2.7) and the homogeneous boundary conditions (2.11). Since λ depends on the vibration frequency ω, equation (2.12) will also be the equation of the natural frequencies of the system.

Since by virtue of (2.9) and (2.10) the parameter λ^2 appears in the arguments of the hyperbolic and trigonometric functions, the characteristic equation (2.12) will be transcendental, yielding an infinite set of values for λ_n^2. It is easily seen that all these values are real, irrespective of the type of boundary-value problem to which the equation corresponds.

The parameters λ_n^2 $(n = 1, 2, 3, \ldots)$ determined by (2.12) are called eigenvalues of the homogeneous boundary-value problem.

Applying the method described here to determine λ^2 thus yields an infinite set of real eigenvalues $\lambda_1^2, \lambda_2^2, \lambda_3^2, \ldots$. To each eigenvalue λ_n^2 there corresponds, in accordance with (2.8), a certain vibration frequency; hence, an infinite set of natural frequencies will correspond to the infinite set of eigenvalues λ_n^2 $(n = 1, 2, 3, \ldots)$.

From (2.8) we obtain:

$$\omega_n^2 = \frac{\lambda^4 + s^4 - r^4}{m^4}. \tag{2.13}$$

Similarly, in accordance with (2.9), a pair of numbers α_n and β_n, which determine the function $X_n(x)$, will correspond to each eigenvalue λ_n^2. We thus obtain an infinite set of functions $X_n(x)$ $(n = 1, 2, 3, \ldots)$ which satisfy all the conditions of the given homogeneous boundary-value problem. These functions are called eigenfunctions (cf. section 4 of Chapter III).

Since the constants $C_1, \ldots, C_4$ are determined by (2.11) up to one common constant factor, each eigenfunction $X_n(x)$ will also be determined up to one constant factor.

2

The eigenfunctions possess the property of orthogonality:

$$\int_0^l X_i X_k \, dx = 0. \quad [i \neq k] \tag{2.14}$$

This can be proved as follows:

Since X_i and X_k are solutions of (2.7), we have:

$$\left.\begin{aligned} (\lambda_i^4 - r^4) X_i &= X_i^{IV} - 2r^2 X_i'', \\ (\lambda_k^4 - r^4) X_k &= X_k^{IV} - 2r^2 X_k'', \end{aligned}\right\} \tag{2.15}$$

where λ_i^4 and λ_k^4 = eigenvalues corresponding to eigenfunctions X_i and X_k

We multiply the first equation (2.15) by X_k, the second by X_i, subtract one from the other, and integrate the resulting equation between $x = 0$ and $x = l$. This yields:

$$(\lambda_i^4 - \lambda_k^4) \int_0^l X_i X_k \, dx = \int_0^l X_i^{IV} X_k \, dx - \int_0^l X_k^{IV} X_i \, dx - 2r^2 \left(\int_0^l X_i'' X_k \, dx - \int_0^l X_k'' X_i \, dx \right). \tag{2.16}$$

Integrating the right-hand side by parts, we obtain:

$$(\lambda_i^4 - \lambda_k^4) \int_0^l X_i X_k \, dx = [X_i''' X_k - X_k''' X_i - X_i'' X_k' + X_k'' X_i' - 2r^2 (X_i' X_k - X_k' X_i)]_0^l, \tag{2.17}$$

where the symbol $[\ ]_0^l$ denotes the difference between the values for $x = l$ and for $x = 0$ of the expression in brackets.

The right-hand side of (2.17) is proportional to the work done in state "i" by the generalized boundary moments and forces M and Q over the boundary displacements corresponding to another state k, and is zero in the case of homogeneous boundary conditions, irrespective of the type of support. Hence:

$$(\lambda_i^4 - \lambda_k^4) \int_0^l X_i X_k \, dx = 0.$$

Since the eigenvalues λ_i^2 and λ_k^2 are distinct when $i \neq k$, it follows that:

$$\lambda_i^4 - \lambda_k^4 \neq 0$$

and therefore:

$$\int_0^l X_i X_k \, dx = 0,$$

which proves the orthogonality of the eigenfunctions.

Exactly as in the case of vibrations of a simple beam (cf. section 4 of Chapter III) all even derivatives of the functions X_i and X_k also possess the property of orthogonality:

$$\int_0^l X_i^{(2m)} X_k^{(2m)} \, dx = 0. \tag{2.18}$$

Here, $X^{(2m)}$ denotes the derivative of order $2m \, (m = 1, 2, 3, \ldots)$. This can be proved for any value of m in the same way as the orthogonality of the eigenfunctions themselves was demonstrated.

3

Some examples of the determination of the eigenfunctions and the natural frequencies of a beam on an elastic single-layer foundation will now be given.

a) Let the beam ends have simple supports. The boundary conditions are then:

at $X'' = 0$ and $X = l$ $\qquad X = X'' = 0. \qquad (2.19)$

The following system of equations in $C_1, \ldots, C_4$ are obtained from these boundary conditions and from (2.10):

$$\begin{aligned} C_2 + C_4 &= 0, \\ \alpha^2 C_2 - \beta^2 C_4 &= 0, \\ C_1 \operatorname{sh} \alpha l + C_2 \operatorname{ch} \alpha l + C_3 \sin \beta l + C_4 \cos \beta l &= 0, \\ C_1 \alpha^2 \operatorname{sh} \alpha l + C_2 \alpha^2 \operatorname{ch} \alpha l - C_3 \beta^2 \sin \beta l - C_4 \beta^2 \cos \beta l &= 0. \end{aligned}$$

The first two equations yield:

$$C_2 = C_4 = 0.$$

The remaining two equations then reduce to:

$$\left.\begin{aligned} \operatorname{sh}\alpha l C_1 + \sin\beta l C_3 = 0, \\ \alpha^2 \operatorname{sh}\alpha l C_1 - \beta^2 \sin\beta l C_3 = 0. \end{aligned}\right\} \qquad (2.20)$$

Equating to zero the determinant of this system, we obtain:

$$(\alpha^2 + \beta^2)\operatorname{sh}\alpha l \sin\beta l = 0$$

or

$$\sin\beta l = 0, \qquad (2.21)$$

whose roots are:

$$\beta_n = \frac{n\pi}{l}. \text{ [where } n \text{ is an integer]} \qquad (2.22)$$

Substitution of (2.22) in the second equation (2.9) yields all the eigenvalues of the given problem:

$$\lambda_n^2 = \frac{n^2\pi^2}{l^2} + r^2. \qquad (2.23)$$

Substituting (2.23) in (2.13), we obtain finally:

$$\omega_n = \sqrt{\frac{1}{m^*}\left[s^4 + 2r^2\left(\frac{n\pi}{l}\right)^2 + \left(\frac{n\pi}{l}\right)^4\right]}. \qquad (2.24)$$

This is a general formula for the natural frequencies of the system. For instance, putting $s = r = 0$, we obtain as a particular case the natural frequency of a simple single-span beam. Putting only $r = 0$, we obtain the frequencies of a beam on a Winkler foundation of modulus k.

Having determined the eigenvalues λ_n^2 and frequencies ω_n, we can find the eigenfunctions X_n. According to (2.21):

$$\sin\beta l = 0.$$

Substitution of this in (2.20) yields:

$$C_1 = 0.$$

Hence, in the case considered three constants vanish:

$$C_1 = C_2 = C_4 = 0. \qquad (2.25)$$

The constant C_3 remains indeterminate.

Substitution of (2.25) in the first equation (2.10) yields the eigenfunctions $X_n(x)$, determined up to a constant factor. Taking this factor as unity, we obtain:

$$X_1 = \sin\frac{\pi x}{l}, \quad X_2 = \sin\frac{2\pi x}{l}, \quad X_3 = \sin\frac{3\pi x}{l}, \ldots \qquad (2.26)$$

b) If the beam end $x = 0$ is simply supported while the end $x = l$ is built in, the boundary conditions will be:

$$\left.\begin{array}{ll} \text{at } x = 0 & X = X'' = 0, \\ \text{at } x = l & X = X' = 0, \end{array}\right\} \tag{2.27}$$

whence:

$$\left.\begin{aligned} C_2 + C_4 &= 0, \\ \alpha^2 C_2 - \beta^2 C_4 &= 0, \\ \operatorname{sh} \alpha l C_1 + \operatorname{ch} \alpha l C_2 + \sin \beta l C_3 + \cos \beta l C_4 &= 0, \\ \alpha \operatorname{ch} \alpha l C_1 + \alpha \operatorname{sh} \alpha l C_2 + \beta \cos \beta l C_3 - \beta \sin \beta l C_4 &= 0. \end{aligned}\right\} \tag{2.28}$$

Thus:

$$\left.\begin{aligned} C_2 = C_4 &= 0, \\ \operatorname{sh} \alpha l C_1 + \sin \beta l C_3 &= 0, \\ \alpha \operatorname{ch} \alpha l C_1 + \beta \cos \beta l C_3 &= 0. \end{aligned}\right\} \tag{2.29}$$

The corresponding characteristic equation is:

$$\beta \operatorname{sh} \alpha l \cos \beta l - \alpha \operatorname{ch} \alpha l \sin \beta l = 0. \tag{2.30}$$

Substitution of:

$$\alpha = \sqrt{\lambda^2 + r^2}, \quad \beta = \sqrt{\lambda^2 - r^2},$$

yields:

$$\sqrt{\lambda^2 - r^2} \operatorname{th} \sqrt{\lambda^2 + r^2}\, l = \sqrt{\lambda^2 + r^2} \operatorname{tg} \sqrt{\lambda^2 - r^2}\, l. \tag{2.31}$$

If r^2 is given, the transcendental equation (2.31) determines all the eigenvalues $\lambda_1^2, \lambda_2^2, \lambda_3^2, \ldots$ of the boundary-value problem considered, whence all the natural frequencies $\omega_1, \omega_2, \omega_3, \ldots$ of the beam are determined by (2.13).

The eigenfunctions $X_n(x)$ corresponding to the eigenvalues λ_n^2 and boundary conditions (2.27) are:

$$X_n(x) = \sin \sqrt{\lambda_n^2 - r^2}\, l \operatorname{sh} \sqrt{\lambda_n^2 + r^2}\, x - \operatorname{sh} \sqrt{\lambda_n^2 + r^2}\, l \sin \sqrt{\lambda_n^2 - r^2}\, x$$
$$(n = 1,\ 2,\ 3, \ldots). \tag{2.32}$$

It is easily seen that these functions satisfy the boundary conditions (2.27).

The functions $X_n(x)$ obtained depend also on the elastic characteristic r. By successively assigning different values to this parameter we obtain different families of eigenfunctions with corresponding eigenvalues, determined for a given r^2 by (2.23); all the functions corresponding to this value of r^2 will be orthogonal. The family of functions obtained for $r^2 = 0$ corresponds to the vibrations of a simple beam with one end simply supported and the other built-in (cf. section 4 of Chapter III).

c) When both beam ends are built in, the boundary conditions are:

$$\begin{aligned} &\text{at } x=0 \qquad X=0, \quad X'=0; \\ &\text{at } x=l \qquad X=0, \quad X'=0. \end{aligned} \tag{2.33}$$

The characteristic equation, obtained in the same way as in the preceding examples, is:

$$\cos\sqrt{\lambda^2-r^2}\,l\,\mathrm{ch}\sqrt{\lambda^2+r^2}\,l-\frac{r^2}{\sqrt{\lambda^4-r^4}}\sin\sqrt{\lambda^2-r^2}\,l\,\mathrm{sh}\sqrt{\lambda^2+r^2}\,l=1. \tag{2.34}$$

The eigenfunctions are:

$$X_n(x)=(\mathrm{ch}\,\alpha_n l-\cos\beta_n l)(\beta_n\,\mathrm{sh}\,\alpha_n x-\alpha_n\sin\beta_n x)-\\-(\beta_n\,\mathrm{sh}\,\alpha_n l-\alpha_n\sin\beta_n l)(\mathrm{ch}\,\alpha_n x-\cos\beta_n x), \tag{2.35}$$

where

$$\alpha_n=\sqrt{\lambda_n^2+r^2}, \qquad \beta_n=\sqrt{\lambda_n^2-r^2}.$$

Putting $r=0$, we obtain the eigenfunctions and eigenvalues of a simple built-in beam.

The above method can also be applied to other boundary conditions.

4

Proceeding from the properties of the general integral of a homogeneous linear differential equation, the following general expression is obtained for the free vibrations of a beam on an elastic single-layer foundation:

$$V(x,t)=\sum_{n=1}^{\infty}X_nT_n=\sum_{n=1}^{\infty}X_n(A_n\sin\omega_n t+B_n\cos\omega_n t), \tag{2.36}$$

where A_n and B_n = integration constants.

This expression can also be written in the following form:

$$V(x,t)=\sum_{n=1}^{\infty}C_nX_n\sin\omega_n(t-\psi_n), \tag{2.37}$$

where the integration constants are now C_n and ψ_n, the latter characterizing the phase shift.

It follows from (2.36) that the elastic line of a freely vibrating beam represents the geometrical sum of an infinite set of curves of the form:

$$X_n(A_n\sin\omega_n t+B_n\cos\omega_n t),$$

which are called principal modes of the transverse vibrations of a beam. Each curve is described by the corresponding function X_n and oscillates at the frequency ω_n. The beam axis therefore changes its shape continuously.

After the beam deflections have been determined in form (2.36), the velocity at each point is obtained from:

$$\frac{\partial V}{\partial t}=\sum_{n=1}^{\infty}X_n\omega_n(A_n\cos\omega_n t-B_n\sin\omega_n t). \tag{2.38}$$

The bending moments and shearing forces are:

$$M(x, t) = -EJ\sum_{n=1}^{\infty} X_n''(A_n \sin\omega_n t + B_n \cos\omega_n t), \tag{2.39}$$

$$Q(x, t) = -EJ\sum_{n=1}^{\infty} X_n'''(A_n \sin\omega_n t + B_n \cos\omega_n t). \tag{2.40}$$

The diagrams of these magnitudes also change their shape continuously. The maximum bending moments and shearing forces are obtained at different beam sections and times.

§ 3. ACTION OF A MOMENTARY IMPULSE

Let an impulse of intensity $p(x)$ per unit length act for an infinitely short time on an elastic beam of length l resting on a single-layer foundation. At the instant at which the load disappears the displacements are still infinitesimal, but the velocities are already finite.

After the load has been removed, the beam will vibrate freely, its deflections being:

$$V(x, t) = \sum_{n=1}^{\infty} X_n(A_n \sin\omega_n t + B_n \cos\omega_n t). \tag{3.1}$$

Assume that the eigenfunctions X_n of the bar and the corresponding frequencies ω_n have already been determined from the boundary conditions. We obtain the coefficients A_n and B_n from the initial conditions, which, according to our assumptions, are:

$$\left.\begin{aligned} V(x, 0) &= 0, \\ \frac{\partial V(x, 0)}{\partial t} &= v_0, \end{aligned}\right\} \tag{3.2}$$

where v_0 = initial velocity at section x.

The first condition (3.2) yields $B_n = 0$. To make use of the second condition, we express the initial velocity of a beam element of length dx in the form:

$$v_0 = \frac{p(x)}{m}, \tag{3.3}$$

where m = reduced mass per unit length of system, which, by the last expression (1.7), is:

$$m = m_1 + m_0 = \frac{\gamma\delta}{g} + \frac{\gamma_0\delta}{g}\int_0^R \psi^2(y)\,dy.$$

Since for $B_n = 0$:

$$\frac{\partial V}{\partial t} = \sum_{n=1}^{\infty} X_n A_n \omega_n \cos\omega_n t,$$

the second condition (3.2) becomes:

$$\frac{p(x)}{m} = \sum_{n=1}^{\infty} A_n \omega_n X_n. \tag{3.4}$$

The determination of the coefficients A_n thus reduces to expanding the function $\frac{p(x)}{m}$ in a series of the eigenfunctions X_n.

We multiply both sides of (3.4) by one of the eigenfunctions and integrate the resulting expression over the entire beam length:

$$\frac{1}{m}\int_0^l p(x) X_k \, dx = \sum_{n=1}^{\infty} \omega_n A_n \int_0^l X_n X_k \, dx. \tag{3.5}$$

Because of the orthogonality of the eigenfunctions, all integrals on the right-hand side vanish for $n \neq k$. The only nonzero integral is:

$$\omega_k A_k \int_0^l X_k^2 \, dx. \tag{3.6}$$

Equation (3.5) thus reduces to:

$$\frac{1}{m}\int_0^l p(x) X_k \, dx = \omega_k A_k \int_0^l X_k^2 \, dx,$$

whence:

$$A_k = \frac{\int_0^l p(x) X_k \, dx}{m\omega_k \int_0^l X_k^2 \, dx}. \tag{3.7}$$

The solution finally obtained is thus:

$$V(x, t) = \frac{1}{m} \sum_{n=1}^{\infty} \frac{\int_0^l p(x) X_n \, dx}{\omega_n \int_0^l X_n^2 \, dx} X_n \sin \omega_n t. \tag{3.8}$$

The bending moments and shearing forces are:

$$\left.\begin{aligned} M(x, t) &= -\frac{EJ}{m}\sum_{n=1}^{\infty}\frac{\int_0^l p(x)X_n\,dx}{\omega_n\int_0^l X_n^2\,dx}X_n''\sin\omega_n t, \\ Q(x, t) &= -\frac{EJ}{m}\sum_{n=1}^{\infty}\frac{\int_0^l p(x)X_n\,dx}{\omega_n\int_0^l X_n^2\,dx}X_n'''\sin\omega_n t. \end{aligned}\right\} \quad (3.9)$$

We thus see that each component of the impulse

$$p_n(x) = mA_n\omega_n X_n$$

causes a simple harmonic motion of mode X_n, frequency ω_n, and amplitude A_nX_n.

Series (3.8) for the deflections converges relatively slowly; series (3.9) for the bending moments and shearing forces converge even more slowly. This may cause considerable difficulties in practice. However, in these expressions no allowance has been made for the damping of the vibrations, which is considerably more rapid for the high-frequency vibrations than for the low-frequency vibrations; most high-frequency vibrations are already completely damped at the instant when the deflection corresponding to the fundamental mode attains its maximum.

As a result, it is sufficient to take the first terms of (3.8) and (3.9) in order to obtain a satisfactory approximation.

Consider, for example, a single-span beam with simple supports resting on an elastic foundation (Figure 143). Let a momentary impulse p, uniformly distributed over the span, be applied to the beam. The eigenfunctions are in this case given by (2.26), and the natural frequencies by (2.24). To determine the coefficients A_n, we substitute (2.26) in (3.7) and obtain:

$$A_n = \frac{p\int_0^l \sin\frac{n\pi}{l}x\,dx}{m\omega_n\int_0^l \sin^2\frac{n\pi}{l}x\,dx} = \frac{4p}{n\pi m\omega_n}. \quad (3.10)$$

Substituting these expressions in (3.8) yields:

$$\begin{aligned} V(x, t) &= \frac{4p}{\pi m}\sum_{n=1}^{\infty}\frac{1}{n\omega_n}\sin\frac{n\pi}{l}x\sin\omega_n t = \\ &= \frac{4p}{\pi m}\left[\frac{1}{\omega_1}\sin\frac{\pi x}{l}\sin\omega_1 t + \frac{1}{3\omega_3}\sin\frac{3\pi x}{l}\sin\omega_3 t + \ldots\right]. \end{aligned} \quad (3.11)$$

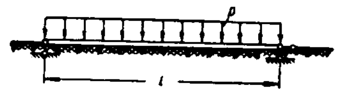


FIGURE 143.

Only odd values of n appear in (3.11), since a uniform impulse will cause the beam to vibrate symmetrically with respect to its center section.

§ 4. FORCED VIBRATIONS OF A BEAM

Consider an external load acting on a beam resting on an elastic foundation:

$$p(x, t) = p(x) f(t). \tag{4.1}$$

Assume that the variation of p is not accompanied by any noticeable variation of the mass of the system, i.e., m remains constant during the motion.

This problem reduces to integrating the nonhomogeneous differential equation (1.6). Its solution is the sum of the general solution of the corresponding homogeneous equation and of a particular solution of the nonhomogeneous equation. The general solution of the homogeneous equation is:

$$V(x, t) = \sum_{n=1}^{\infty} X_n (A_n \sin \omega_n t + B_n \cos \omega_n t), \tag{4.2}$$

where A_n and B_n are found by expanding the given initial values of V and $\frac{\partial V}{\partial t}$ in series of the eigenfunctions X_n, as shown in the preceding section for the case of a momentary impulse.

To determine a particular solution of (1.6), we expand $p(x)$ in a series of the eigenfunctions X_n:

$$p(x) = \sum_{n=1}^{\infty} C_n X_n. \tag{4.3}$$

The coefficients C_n can be determined by multiplying both sides of (4.3) by X_k and integrating over the entire beam length. Because of the orthogonality of the eigenfunctions, we obtain:

$$C_k = \frac{\int_0^l p(x) X_k \, dx}{\int_0^l X_k^2 \, dx}. \tag{4.4}$$

If the external load also includes concentrated forces P, the integral in the numerator of (4.4) is to be understood as a Stieltjes integral. We can therefore rewrite (4.4) as follows:

$$C_k = \frac{\int_0^l p(x) X_k \, dx + \sum P\overline{X}_k}{\int_0^l X_k^2 \, dx}, \tag{4.5}$$

where $\overline{X}_k$ = values of functions X_k at points of application of forces P.

Each function X_n can be considered as an elastic line induced by a distributed static load of intensity $\omega_n^2 m X_n$. The static load $C_n X_n$ will obviously induce an elastic line whose ordinates are $\frac{C_n}{\omega_n^2 m} X_n$. A dynamic load causes at time t the displacements*

$$V(x, t) = \omega \int_0^t V_{st}(u) \sin \omega (t-u) du, \tag{4.6}$$

In the case considered the static load is $C_n X_n f(t)$. Hence:

$$V_n(x, t) = \frac{C_n}{m\omega_n} X_n \int_0^t f(u) \sin \omega_n (t-u) \, du. \tag{4.7}$$

Hence by (4.3):

$$V(x, t) = \sum_{n=1}^{\infty} \frac{C_n}{m\omega_n} X_n \int_0^t f(u) \sin \omega_n (t-u) \, du. \tag{4.8}$$

The general solution of (1.6) is the sum of (4.2) and (4.8):

$$V(x, t) = \sum_{n=1}^{\infty} X_n \left[A_n \sin \omega_n t + B_n \cos \omega_n t + \frac{C_n}{m\omega_n} \int_0^t f(u) \sin \omega_n (t-u) \, du \right]. \tag{4.9}$$

The bending moments and shearing forces are calculated by the known formulas of strength of materials:

$$M = -EJ \frac{\partial^2 V}{\partial x^2}, \quad Q = -EJ \frac{\partial^3 V}{\partial x^3}; \tag{4.10}$$

the velocity is found by differentiating (4.9) with respect to t:

$$v = \frac{\partial V}{\partial t} = \sum_{n=1}^{\infty} X_n \left[A_n \omega_n \cos \omega_n t - B_n \omega_n \sin \omega_n t \;\; \frac{C_n}{m} \int_0^t f(u) \cos \omega_n (t-u) \, du \right]. \tag{4.11}$$

* See for instance, Rabinovich, I. M., Kurs stroitel'noi mekhaniki (A Course in Structural Mechanics), part II. 1954.

§ 5. DYNAMICAL ANALYSIS OF BEAMS CONSIDERED AS SYSTEMS WITH FINITE NUMBERS OF DEGREES OF FREEDOM

From the viewpoint of structural dynamics, the beams considered by us represent systems with infinitely large numbers of degrees of freedom. The deformed axis of such a system can acquire an infinitely large number of different shapes under the action of the various static and dynamic loads. The exact dynamical analysis of elastic beams, which takes into account the entire frequency spectrum, thus leads in general to an infinite series of the eigenfunctions of the given boundary-value problem.

However, it can frequently be assumed that the beam considered has a finite number of degrees of freedom. If a beam resting on an elastic foundation is so rigid that it does not become bent, it can be considered as a system with two degrees of freedom. Elastic beams can also be considered as systems with finite numbers of degrees of freedom. In this case some basic forms are selected from the infinitely large number of forms which the elastic line of the beam may assume, and only these forms are considered in the calculations. Any function approximating the elastic line of the beam and satisfying the geometrical boundary conditions can be taken as vibrational mode. This considerably simplifies the solution of the dynamical problem while being satisfactory for practical needs.

1

We consider first the vibrations of a rigid beam on an elastic single-layer foundation (Figure 144). Let the deflections of the beam be:

$$V = CT(t). \tag{5.1}$$

The equilibrium conditions of the beam are obtained by equating to zero the work done by all the forces acting on the beam over the virtual displacement $\overline{V} = 1$. The inertia forces per unit beam length are:

$$m_1 = \frac{\partial^2 V}{\partial t^2} = m_1 C T''. \tag{5.2}$$

In addition to these forces, there act on the beam also the reactions q and Q^{Φ} of the elastic foundation.

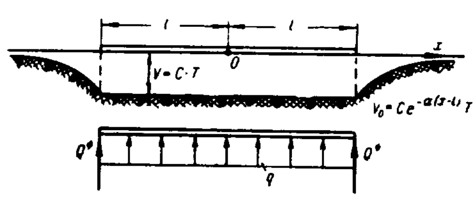


FIGURE 144.

In the general case, when allowance is made for the inertia of the foundation, the distributed reactions q are by (1.5):

$$q = -2t\frac{\partial^2 V}{\partial x^2} + kV + m_0\frac{\partial^2 V}{\partial t^2}. \tag{5.3}$$

Substitution of (5.1) yields:

$$q = kCT + m_0 CT''. \tag{5.4}$$

To determine the fictitious reactions Q^{Φ}, it is necessary to calculate the work done by the normal and shearing stresses, and also by the inertia forces in the elastic foundation beyond the beam ends. The virtual displacements to the right of the beam are:

$$\bar{V}_0 = e^{-\alpha(x-l)}.$$

We then obtain:

$$Q^{\Phi} = \left(2\alpha tT + m_0\frac{\alpha t}{k}T''\right)C = \left(2\alpha tT + \frac{m_0}{2\alpha}T''\right)C, \tag{5.5}$$

where:

$$\left.\begin{aligned} k &= \frac{E_0\delta}{1-\nu_0^2}\int_0^H \psi'^2(y)\,dy,\\ t &= \frac{E_0\delta}{4(1+\nu_0)}\int_0^H \psi^2(y)\,dy,\\ \alpha &= \sqrt{\frac{k}{2t}}. \end{aligned}\right\} \tag{5.6}$$

Equation (5.5) for the concentrated reactions, valid for the dynamical problem, differs from (5.7) of Chapter II by the presence of a second term representing the inertia forces in the elastic foundation.

Equating to zero the work done by all the forces acting on the beam, we obtain:

$$(2lk + 4\alpha t)T + \left(2lm_0 + 2lm_1 + 2m_0\frac{\alpha t}{k}\right)T'' = 0, \tag{5.7}$$

or

$$T'' + \frac{k\left(1+\frac{1}{\alpha l}\right)}{m_1\left(1+\frac{m_0}{m_1}+\frac{m_0}{m_1}\frac{1}{2\alpha l}\right)}T = 0. \tag{5.8}$$

Thus, a rigid beam on an elastic foundation performs a simple harmonic motion at a frequency of

$$\omega = \sqrt{\frac{k}{m_1}\frac{1+\frac{1}{\alpha l}}{1+\frac{m_0}{m_1}+\frac{m_0}{m_1}\frac{1}{2\alpha l}}}. \tag{5.9}$$

The vertical displacement of the beam at any instant is

$$V = C_1 \sin \omega t + C_2 \cos \omega t, \tag{5.10}$$

where C_1 and C_2 = constants determined from initial conditions.

2

We consider as a second example the free vibrations of an elastic beam of length $2l$ resting on a single-layer foundation (Figure 145). Let the motion be symmetrical with respect to the section $x = 0$. The elastic line of the beam is then approximately defined as follows:

$$X(x) = \sum_{i=0}^{n} A_i \cos \frac{i\pi x}{2l} \qquad (i = 0,\ 1,\ 3,\ 5, \ldots, n). \tag{5.11}$$

The first term of this series ($i = 0$) corresponds to the displacements of a rigid beam; the other terms correspond to a symmetrical bending. It can be seen that the function $X(x)$ satisfies the geometrical conditions of the problem and one of the statical boundary conditions ($M = 0$ at $x = \pm l$). The second boundary conditions ($Q = 0$ at $x = \pm l$) is not satisfied.

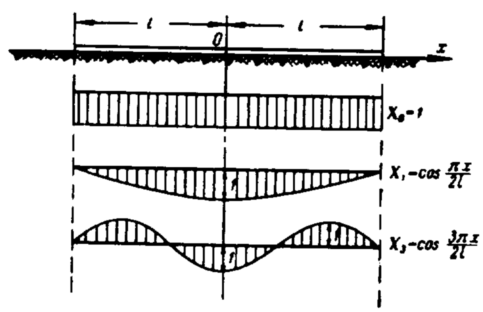


FIGURE 145.

To find the natural frequencies ω_i of the beam, corresponding to modes (5.11), we obtain the equilibrium conditions for the beam by means of the principle of virtual displacements. Taking (1.7) into account, (2.5) can be written as follows:

$$EJX^{IV} - 2tX'' + (k - m\omega^2) X = 0. \tag{5.12}$$

Substitution of (5.11) yields:

$$\sum_{i=0}^{n} A_i \left[EJ \left(\frac{i\pi}{2l}\right)^4 + 2t \left(\frac{i\pi}{2l}\right)^2 + (k - m\omega^2) \right] \cos \frac{i\pi x}{2l} = 0. \tag{5.13}$$

For each value of i, the first term corresponds to bending of the beam; the second and third terms containing the coefficients k and t depend on the reactions of the elastic foundation, distributed over the bottom of the beam; the last term depends on the inertia forces.

In addition to these loads, concentrated reactions Q^{Φ} will also act on the ends of the beam when the latter is forced into the soil like a rigid body. These reactions are determined from (5.5)*:

$$Q^{\Phi} = \left(2\alpha t - \frac{m_0}{2\alpha}\omega^2\right) A_0. \tag{5.14}$$

The following system of algebraic equations is obtained by calculating the work done by all these forces over each virtual displacement of the beam:

$$\left.\begin{aligned}
&\int_{-l}^{l}\sum_{i=0}^{n}(k - m\omega^2) A_i \cos\frac{i\pi x}{2l}\,dx + 2Q^{\Phi} = 0,\\
&\int_{-l}^{l}\sum_{i=0}^{n}\left[EJ\left(\frac{i\pi}{2l}\right)^4 + 2t\left(\frac{i\pi}{2l}\right)^2 + k - m\omega^2\right]\times\\
&\qquad\qquad\times A_i\cos\frac{i\pi x}{2l}\cos\frac{\pi x}{2l}\,dx = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\int_{-l}^{l}\sum_{i=0}^{n}\left[EJ\left(\frac{i\pi}{2l}\right)^4 + 2t\left(\frac{i\pi}{2l}\right)^2 + k - m\omega^2\right]\times\\
&\qquad\qquad\times A_i\cos\frac{i\pi x}{2l}\cos\frac{n\pi x}{2l}\,dx = 0\\
&\qquad\qquad (i = 0, 1, 3, 5, 7, \ldots, n),
\end{aligned}\right\} \tag{5.15}$$

or, after the corresponding integrations are performed:

$$\left.\begin{aligned}
&\left[k\left(1 + \frac{1}{\alpha l}\right) - \left(m_1 + m_0 + m_0\frac{1}{2\alpha l}\right)\omega^2\right]A_0 +\\
&\qquad\qquad + \sum_{i=1}^{n} A_i \frac{2}{i\pi}(k - m\omega^2)(-1)^{\frac{i-1}{2}} = 0,\\
&\frac{2}{\pi}(k - m\omega^2) A_0 + \frac{1}{2}\left(EJ\frac{\pi^4}{16l^4} +\right.\\
&\qquad\qquad \left. + 2t\frac{\pi^2}{4l^2} + k - m\omega^2\right) A_1 = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{2}{n\pi}(k - m\omega^2)(-1)^{\frac{n-1}{2}} A_0 +\\
&\qquad + \frac{1}{2}\left(EJ\frac{n^4\pi^4}{16l^4} + 2t\frac{n^2\pi^2}{4l^2} + k - m\omega^2\right) A_n = 0\\
&\qquad\qquad (i = 1, 3, 5, 7, \ldots, n).
\end{aligned}\right\} \tag{5.16}$$

The $\frac{n+3}{2}$ homogeneous algebraic equations (5.16) become identitites for $A_0 = A_1 = A_3 = \ldots = A_n = 0$. This trivial solution corresponds to the case when the beam does not vibrate. A nontrivial solution is obtained by equating to zero the determinant of the coefficients of the constants A_i. This yields an

* It is assumed that the elastic foundation beyond the beam ends performs harmonic motion of frequency ω.

equation of order $\frac{n+3}{2}$ in the unknown ω^2, whose solution gives all the natural frequencies ω_i corresponding to the vibrational modes (5.11).

The vibration frequency ω_0 of a rigid beam, given by (5.9), is a particular case. This solution is also applicable to the natural vibrations of a beam on simple supports. In this case, the first term $(i=0)$ in (5.11) and the corresponding terms in (5.16) are discarded. Since in practice only the lowest frequencies are of interest, it is sufficient in the general case of a beam lying freely on the foundation to take only the first two or three terms in (5.11). The low frequencies are therefore obtained from a quadratic or cubic equation.

It was assumed in the above examples that the beam vibrates symmetrically with respect to the section $x=0$. In the general case, due to the orthogonality of the symmetrical and antisymmetrical modes of vibration, the problem can be divided into two independent parts corresponding respectively to the symmetrical and to the antisymmetrical modes. The frequencies corresponding to the antisymmetrical modes can be determined by the same method as described above.

3

Let a beam with free ends be subjected to the action of a momentary impulse of intensity $p(x)$ per unit length.

The initial conditions are:

$$\left.\begin{aligned} &V(x,\ 0)=0,\\ &\frac{\partial V(x,\ 0)}{\partial t}=\frac{p(x)}{m}. \end{aligned}\right\} \tag{5.17}$$

From (5.11) and the first condition (5.17), the beam deflections are approximately given by:

$$V(x,\ t)=\sum_{i=0}^{n}A_i\cos\frac{i\pi x}{2l}\sin\omega_i t, \tag{5.18}$$

where the vibration frequency ω_i is determined from (5.16).

The second condition (5.17) yields:

$$\frac{p(x)}{m}=\sum_{i=0}^{n}A_i\omega_i\cos\frac{i\pi x}{2l}. \tag{5.19}$$

The coefficients A_i are determined by expanding the function $\frac{p(x)}{m}$ in a series of $\cos\frac{i\pi x}{2l}$ $(i=0,\ 1,\ 3,\ 5,\dots)$. Multiplying in turn both sides of (5.19) by $\cos\frac{i\pi x}{2l}$ $(i=0,\ 1,\ 3,\ 5,\dots)$ and integrating the resulting expressions from $x=0$

to = , we obtain:

$$\left.\begin{aligned}
&A_0\omega_0 + \sum_{i=1}^{n} A_i\omega_i \frac{2}{i\pi}(-1)^{\frac{i-1}{2}} = \frac{1}{ml}\int_0^l p(x)\,dx,\\
&\frac{4}{\pi}A_0\omega_0 + A_1\omega_1 = \frac{2}{ml}\int_0^l p(x)\cos\frac{\pi x}{2l}\,dx,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{4}{n\pi}A_0\omega_0(-1)^{\frac{n-1}{2}} + A_n\omega_n = \frac{2}{ml}\int_0^l p(x)\cos\frac{n\pi x}{2l}\,dx\\
&\qquad (i = 1, 3, 5, 7, \ldots, n),
\end{aligned}\right\} \qquad (5.20)$$

from which all coefficients A_i in (5.18) can be found.

After the beam deflections have been determined in the form (5.18), the bending moments and shearing forces corresponding to $i \geqslant 1$ can be calculated by means of (4.10). The forces and moments corresponding to rigid-body motion of the beam (the first term in (5.18)) are determined by the reactions of the elastic foundation which can be found from (5.4) and (5.5) when A_0 is known. The bending moments and shearing forces of the beam can then be calculated by the usual methods.

§ 6. DIFFERENTIAL EQUATION OF VIBRATIONS OF A PLATE RESTING ON AN ELASTIC SINGLE-LAYER FOUNDATION

During bending vibrations, a plate resting on an elastic foundation can be considered as being in static equilibrium, the plate elements being acted upon by inertia forces $\left(-m_1\frac{\partial^2 w}{\partial t^2}\right)$ in addition to a distributed load $p^*(x, y, t)$ The differential equation of plate vibrations can thus be obtained from (1.1) of Chapter III in the form:

$$D\nabla^2\nabla^2 w(x, y, t) = p^*(x, y, t) - m_1\frac{\partial^2 w(x, y, t)}{\partial t^2}, \qquad (6.1)$$

where m_1 = mass per unit plate area; $p^*(x, y, t)$ = distributed load, consisting of given forces $p(x, y, t)$ per unit area and of reactions $q(x, y, t)$ of elastic foundation:

$$p^*(x, y, t) = p(x, y, t) - q(x, y, t). \qquad (6.2)$$

The inertia of the elastic foundation must be taken into account when calculating the reactions $q(x, y, t)$.

To determine the latter, we cut out from the elastic foundation an elementary column of cross section $dx = 1$, $dy = 1$ and consider its equilibrium conditions, applying Lagrange's principle of virtual displacements (cf. section 6 of Chapter I).

Assume that no horizontal displacements occur in the elastic foundation, and that the z-distribution of the vertical displacements is given by a single function $\psi(z)$ *:

$$\left.\begin{aligned} u(x, y, z, t) &= 0, \\ v(x, y, z, t) &= 0, \\ w(x, y, z, t) &= w(x, y, t)\,\psi(z). \end{aligned}\right\} \tag{6.3}$$

Taking into account the inertia of the elastic foundation, the generalized equilibrium conditions of the elementary column are:

$$\int_0^H \frac{\partial \tau_{zx}}{\partial x}\psi\,dz - \int_0^H \sigma_z \psi'\,dz + \int_0^H \frac{\partial \tau_{zy}}{\partial y}\psi\,dz - \int_0^H \bar{m}_0 \frac{\partial^2 w(x, y, t)}{\partial t^2}\psi^2 dz + q(x, y, t) = 0, \tag{6.4}$$

where $q(x, y, t)$ = load applied to unit area of elastic-foundation surface, $\bar{m}_0 = \frac{\gamma_0}{g}$ = mass per unit volume of elastic foundation, γ_0 = specific weight of soil, g = gravitational acceleration, and H = thickness of compressible layer.

By substituting in (6.4) the values of the normal and shearing stresses of the elastic foundation σ_z, τ_{zx}, τ_{zy} (determined from (6.2) of Chapter I), we obtain after some transformations:

$$-2t\nabla^2 w(x, y, t) + kw(x, y, t) + m_0 \frac{\partial^2 w(x, y, t)}{\partial t^2} = q(x, y, t), \tag{6.5}$$

where

$$\left.\begin{aligned} k &= \frac{E_0}{1-\nu_0^2}\int_0^H \psi'^2(z)\,dz, \\ t &= \frac{E_0}{4(1+\nu_0)}\int_0^H \psi^2(z)\,dz, \\ m_0 &= \bar{m}_0\int_0^H \psi^2(z)\,dz. \end{aligned}\right\} \tag{6.6}$$

The partial differential equation (6.5) describes the vibrations of the elastic foundation, due to a load $q(x, y, t)$. This equation can be considered together with (6.1), since the plate deflections are equal to the vertical displacements of the surface of the elastic foundation beneath the plate, and since the load $q(x, y, t)$ on the foundation equals the reactions of the elastic foundation on the plate.

Eliminating $q(x, y, t)$ between (6.1) and (6.5), we obtain:

$$\nabla^2\nabla^2 w - 2r^2\nabla^2 w + s^4 w + m^* \frac{\partial^2 w}{\partial t^2} = \frac{p}{D}, \tag{6.7}$$

* The function $\psi(z)$ is selected in such a way that $\psi(0) = 1$.

where:

$$
\left.
\begin{aligned}
s^4 &= \frac{k}{D} = \frac{E_0}{(1-\nu_0^2)D}\int_0^H \psi'^2(z)\,dz,\\
r^2 &= \frac{t}{D} = \frac{E_0}{4(1+\nu_0)D}\int_0^H \psi^2(z)\,dz,\\
m^* &= \frac{m_1+m_0}{D} = \left(\frac{\gamma h}{g} + \frac{\gamma_0}{g}\int_0^H \psi^2(z)\,dz\right)\frac{1}{D}
\end{aligned}
\right\} \qquad (6.8)
$$

Here, D = flexural rigidity of plate, γ and γ_0 = specific weights of plate material and of soil respectively, h = plate thickness, g = gravitational acceleration, and:

$$E_0 = \frac{E_s}{1-\nu_s^2}, \qquad \nu_0 = \frac{\nu_s}{1-\nu_s}. \qquad (6.9)$$

In the case of free vibrations, when no external load $p(x, y, t)$ acts, (6.7) reduces to:

$$\nabla^2\nabla^2 w - 2r^2\nabla^2 w + s^4 w = -m^*\frac{\partial^2 w}{\partial t^2}. \qquad (6.10)$$

§ 7. APPROXIMATIVE ANALYSIS OF AN INFINITE PLATE IN THE CASE OF CONCENTRATED IMPACT*

Consider an infinite plate resting on an elastic single-layer foundation (Figure 146). Let the concentrated force $P(t)$, shown in Figure 147 as a function of t, be applied suddenly at some point of the plate, inducing a vibrational motion of the plate. The problem is to determine the stresses and strains of the plate during impact ($0 \leqslant t \leqslant \tau$) and during the ensuing free vibrations of the plate ($\tau \leqslant t < \infty$).

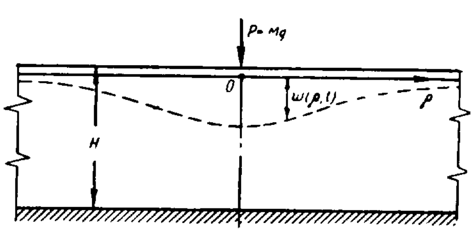


FIGURE 146.

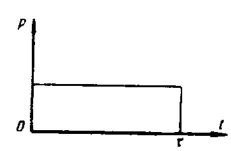


FIGURE 147.

We introduce a system of polar coordinates. Let the origin be at the point of application of the force. Obviously, the plate deflections will depend

* The calculation given in this section was performed by E. I. Silkin at the Institute of Mechanics of the USSR Academy of Sciences.

only on the space coordinate ρ and on the time t, i.e.:

$$w = w(\rho,\ t).$$

The fundamental equation describing the free vibrations of the plate is:

$$\nabla_\rho^2\nabla_\rho^2 w - 2r^2\nabla_\rho^2 w + s^4 w = -m^* \frac{\partial^2 w}{\partial t^2}, \tag{7.1}$$

where

$$\left.\begin{aligned} \nabla_\rho^2 &= \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}, \\ \nabla_\rho^2\nabla_\rho^2 &= \frac{\partial^4}{\partial\rho^4} + \frac{2}{\rho}\frac{\partial^3}{\partial\rho^3} - \frac{1}{\rho^2}\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^3}\frac{\partial}{\partial\rho}. \end{aligned}\right\} \tag{7.2}$$

1. Solution of the fundamental differential equation

We present the solution of the homogeneous equation corresponding to (7.1) in the form:

$$w(\rho,\ t) = W(\rho)\,T(t), \tag{7.3}$$

where $W(\rho)$ = function of ρ only, and $T(t)$ = function of t only.

Substitution of (7.3) in (7.1) yields:

$$\frac{\nabla^2\nabla^2 W - 2r^2\nabla^2 W}{W} = -\frac{m^* T'' + s^4 T}{T}. \tag{7.4}$$

Since the two sides of (7.4) are functions of different variables, each of them must be equal to the same constant in order that the equation be satisfied in the general case:

$$\frac{\nabla^2\nabla^2 W - 2r^2\nabla^2 W}{W} = -\frac{m^* T'' + s^4 T}{T} = \lambda^4, \tag{7.5}$$

where λ^4 = parameter to be determined.

It follows from (7.5) that:

$$m^* T'' + (s^4 + \lambda^4)\,T = 0, \tag{7.6}$$

$$\nabla^2\nabla^2 W - 2r^2\nabla^2 W - \lambda^4 W = 0. \tag{7.7}$$

Putting

$$\frac{s^4 + \lambda^4}{m^*} = \omega^2, \tag{7.8}$$

(7.6) then becomes:

$$T'' + \omega^2 T = 0, \tag{7.9}$$

whose solution is:

$$T = D_1 \sin \omega t + D_2 \cos \omega t, \tag{7.10}$$

where D_1 and D_2 = integration constants.

Equation (7.7) determines the mode of the free vibrations of the plate. We assume a solution of the form:

$$\nabla^2 W = nW. \tag{7.11}$$

Substitution of (7.11) in (7.7) yields:

$$n^2 - 2r^2 n - \lambda^4 = 0,$$

whence:

$$n = r^2 \pm \sqrt{r^4 + \lambda^4}. \tag{7.12}$$

It is seen from (7.11) and (7.12) that the fourth-order differential equation (7.7) is equivalent to the two second-order differential equations:

$$\left.\begin{aligned} \nabla^2 W + (\sqrt{r^4+\lambda^4} - r^2)\, W = 0, \\ \nabla^2 W - (\sqrt{r^4+\lambda^4} + r^2)\, W = 0. \end{aligned}\right\} \tag{7.13}$$

These two equations can be reduced to the same form:

$$\frac{d^2 W}{dx^2} + \frac{1}{x}\frac{dW}{dx} + W = 0. \tag{7.14}$$

by putting in the first

$$x = \rho \sqrt{\sqrt{r^4+\lambda^4} - r^2}$$

and in the second

$$x = i\rho \sqrt{\sqrt{r^4+\lambda^4} + r^2},$$

Equations (7.13) are thus reduced to two zero-order Bessel equations of a real and an imaginary argument respectively.

Considering again the variable ρ, the solution of (7.7) can be represented as Bessel functions of the first and second kinds of a real and an imaginery arguments*:

$$\begin{aligned} W(\rho) = A_1 J_0\left(\rho\sqrt{\sqrt{r^4+\lambda^4} - r^2}\right) + A_2 Y_0\left(\rho\sqrt{\sqrt{r^4+\lambda^4} - r^2}\right) + \\ + A_3 I_0\left(\rho\sqrt{\sqrt{r^4+\lambda^4} + r^2}\right) + A_4 K_0\left(\rho\sqrt{\sqrt{r^4+\lambda^4} + r^2}\right), \end{aligned} \tag{7.15}$$

where J_0 and Y_0 = Bessel functions of a real argument, of the first and second kind respectively; I_0 and K_0 = Bessel functions, of an imaginary argument, of the first and second kind respectively (modified Bessel functions); A_1, A_2, A_3, A_4 = integration constants.

By (7.3), (7.10), and (7.15), the general solution of (7.1) is:

$$w(\rho, t) = [D_1 \sin \omega t + D_2 \cos \omega t]\,[A_1 J_0 + A_2 Y_0 + A_3 I_0 + A_4 K_0]. \tag{7.16}$$

* In section 2 of Chapter IV the solution of a similar equation was presented in a different form. See also section 7 of Chapter I and section 6 of Chapter V.

2. Determining the integration constants from the initial and boundary conditions

To determine the six integration constants in (7.16) it is necessary to consider the initial and boundary conditions for the plate. Physical considerations impose the following boundary conditions on the function $w(\rho, t)$:

$$\left.\begin{aligned} &\text{at } \rho\to 0: && w \neq \infty; \\ &\text{at } \rho\to\infty: && w\to 0. \end{aligned}\right\} \tag{7.17}$$

Since for $\rho = 0$ the functions Y_0 and K_0 tend to infinity, while for $\rho\to\infty$ the function I_0 tends to infinity (Figures 28, 148, and 149), we must have:

$$A_2 = A_3 = A_4 = 0, \tag{7.18}$$

Let no deflections occur before impact:

$$\text{at } t = 0: \qquad w = 0. \tag{7.19}$$

Hence, $D_2 = 0$, and (7.16) reduces to:

$$w(\rho, t) = C\sin\omega t\cdot J_0\left(\rho\sqrt{\sqrt{r^4+\lambda^4}-r^2}\right), \tag{7.20}$$

where C is a constant. To determine its value, let the striking body of mass M have a velocity V_0 at the instant of impact ($t = 0$) If the rigidity of the plate is small, we can write:

$$\text{at } t = 0 \text{ and } \rho = 0: \qquad \frac{\partial w}{\partial t} = V_0. \tag{7.21}$$

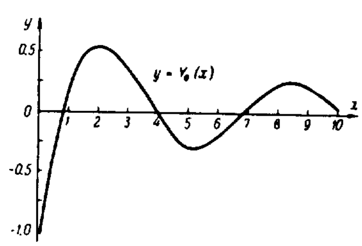


FIGURE 148.

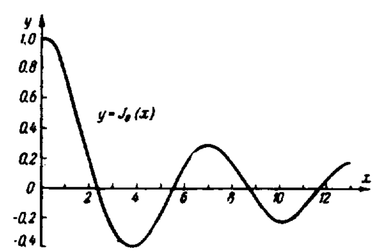


FIGURE 149.

This equation states that the velocities of the striking body and the plate are equal at the point and time of impact.

Since

$$\frac{\partial w}{\partial t} = C\omega\cos\omega t\cdot J_0\left(\rho\sqrt{\sqrt{r^4+\lambda^4}-r^2}\right)$$

and

$$J_0(0) = 1,$$

we can rewrite (7.21) as follows:

$$C\omega = V_0,$$

whence:

$$C = \frac{V_0}{\omega}. \tag{7.22}$$

Substitution of (7.22) in (7.20) yields:

$$w(\rho,\ t) = \frac{V_0}{\omega} \sin \omega t \cdot J_0\left(\rho \sqrt{\sqrt{r^4 + \lambda^4} - r^2}\right). \tag{7.23}$$

In the case of a massive plate, the coefficient C can be found by equating the momentums of the striking body and the plate at the time of impact:

$$\begin{aligned} \frac{PV_0}{g} = \frac{P}{g} C\omega \cos \omega t \cdot J_0(0)\Big|_{t=0} + \\ + \int_0^{\mu_1} 2\pi m C\omega \cos \omega t \cdot J_0\left(\rho \sqrt{\sqrt{r^4 + \lambda^4} - r^2}\right) \rho\, d\rho\Big|_{t=0} + \\ + \sum_{n=1} \int_{\mu_n}^{\mu_{n+1}} 2\pi m C\omega \cos \omega t \cdot J_0\left(\rho \sqrt{\sqrt{r^4 + \lambda^4} - r^2}\right) \rho\, d\rho\Big|_{t=0}, \end{aligned} \tag{7.24}$$

where m = mass per unit area of plate and elastic foundation, determined from the last expression (6.8):

$$m = m^* D = \frac{\gamma h}{g} + \frac{\gamma_0}{g} \int_0^H \psi^2(z)\, dz; \tag{7.25}$$

and $\mu_1, \mu_2, \ldots, \mu_n$ = roots of the function:

$$J_0\left(\rho \sqrt{\sqrt{r^4 - \lambda^4} - r^2}\right),$$

plotted in Figure 149.

The absolute value of the integrals has to be taken in (7.24). The summation in the last term is extended over all half-waves of the function:

$$J_0\left(\rho \sqrt{\sqrt{r^4 + \lambda^4} - r^2}\right),$$

which lie inside the zone of motion of the plate, i.e., that part of the plate which acquires a velocity at the time of impact. The boundaries of this zone can be determined experimentally. Assume as a first approximation that the influence of the impact extends only over a distance corresponding to the first half-wave of J_0. We then obtain for $t = 0$:

$$\frac{PV_0}{g} = \frac{P}{g} C\omega + 2\pi C\omega m \int_0^{\mu_1} \rho J_0\left(\rho \sqrt{\sqrt{r^4 + \lambda^4} - r^2}\right) d\rho. \tag{7.26}$$

The integral is equal to:

$$\int_0^{\mu_1} \rho J_0\left(\rho \sqrt{\sqrt{r^4 + \lambda^4} - r^2}\right) d\rho = \frac{1}{(\sqrt{r^4 + \lambda^4} - r^2)} \int_0^{\mu_1} x J_0(x)\, dx = \frac{x J_1(x)}{(\sqrt{r^4 + \lambda^4} - r^2)}\Bigg|_0^{\mu_1}$$

or, since the first root of $J_0(x)$ is $\mu_1 = 2.4048$ and $J_1(\mu_1) = 0.5191$,

$$\int_0^{\mu_1} \rho J_0\left(\rho\sqrt{\sqrt{r^4+\lambda^4}-r^2}\right) d\rho = \frac{1{,}250}{\sqrt{r^4+\lambda^4}-r^2}.$$

Substituting this value in (7.26), we obtain:

$$C = \frac{PV_0(\sqrt{r^4+\lambda^4}-r^2)}{\omega\,[P(\sqrt{r^4+\lambda^4}-r^2)+2{,}5\pi mg]}. \qquad (7.27)$$

Substitution of (7.27) in (7.20) then yields:

$$w(\rho,\ t) = \frac{PV_0(\sqrt{r^4+\lambda^4}-r^2)}{\omega\,[P(\sqrt{r^4+\lambda^4}-r^2)+2{,}5\pi mg]}\sin\omega t\cdot J_0\left(\rho\sqrt{\sqrt{r^4+\lambda^4}-r^2}\right). \qquad (7.28)$$

3. Obtaining the parameter λ determining the vibration frequency ω of the plate

The parameter λ appearing in (7.23) and (7.28) is related to the vibration frequency ω of the plate by (7.8), and is determined by considering the equilibrium condition of a cylindrical element cut out of the plate near the point of impact (Figure 150).

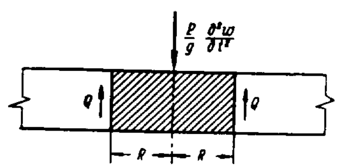


FIGURE 150.

Neglecting the foundation reactions and inertia, the equilibrium condition of this element is approximately given by:

$$2\pi RQ = \frac{P}{g}\frac{\partial^2 w}{\partial t^2}, \qquad (7.29)$$

where R = radius of cut-out element and Q = shearing force acting along its edge:

$$Q = -D\frac{\partial}{\partial\rho}\left(\frac{\partial^2 w}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial w}{\partial\rho}\right). \qquad (7.30)$$

Introducing the coordinate R, defined by:

$$R = \rho\sqrt{\sqrt{r^4+\lambda^4}-r^2}. \qquad (7.31)$$

we can rewrite (7.30) as follows:

$$Q = -D\left(\sqrt{r^4+\lambda^4} - r^2\right)^{3/2}\left(\frac{\partial^3 w}{\partial R^3} - \frac{1}{R^2}\frac{\partial w}{\partial R} + \frac{1}{R}\frac{\partial^2 w}{\partial R^2}\right). \tag{7.32}$$

Substitution of (7.20) yields:

$$Q = -D\left(\sqrt{r^4+\lambda^4} - r^2\right)^{3/2} C \sin \omega t \left[J_0'''(R) - \frac{1}{R^2}J_0'(R) + \frac{1}{R}J_0''(R)\right]. \tag{7.33}$$

Since:

$$J_0' = -J_1, J_0'' = \frac{J_1}{R} - J_0, \quad J_0''' = \frac{RJ_0 - J_1}{R^2} - \frac{J_1}{R^2} + J_1,$$

(7.33) can be rewritten as follows:

$$Q = -D\left(\sqrt{r^4+\lambda^4} - r^2\right)^{3/2} C \sin \omega t \cdot J_1(R). \tag{7.34}$$

The function $J_1(R)$ has been plotted in Figure 151. It is seen that (7.34) is not valid at $\rho = 0$. We therefore exclude the zone inside the radius $R = 1.8$ from our consideration. [For $R = 1.8$], we then obtain:

$$Q = -0.582\, D\left(\sqrt{r^4+\lambda^4} - r^2\right)^{3/2} C \sin \omega t. \tag{7.35}$$

Substituting (7.35) and (7.20) in (7.29), and taking (7.8) and (6.8) into account, we obtain:

$$\left(\sqrt{r^4+\lambda^4} - r^2\right)^{3/2} = \frac{P\left(s^4+\lambda^4\right)}{19.4\left[\gamma h + \gamma_0 \int\limits_0^H \psi^2(z)\, dz\right]}. \tag{7.36}$$

After λ^4 has been determined from (7.36), the frequency of the plate vibrations during the impact can be calculated by (7.8).

4. Free plate vibrations

At the instant at which the impact ends ($t = \tau$) the plate has already acquired finite displacements and velocities, and it continues thereafter to vibrate freely. The displacement and velocity at the beginning of the free vibrations are, by (7.20):

$$\left.\begin{aligned} w &= C \sin \omega\tau J_0, \\ \frac{\partial w}{\partial t} &= C\omega \cos \omega\tau \cdot J_0. \end{aligned}\right\} \tag{7.37}$$

Taking into account the boundary conditions at the origin of coordinates and at infinity, the solution of (7.1) is:

$$\begin{aligned} w(\rho, t) &= \left[D_1 \sin \omega^*(t-\tau) + D_2 \cos \omega^*(t-\tau)\right] A_1 J_0^* = \\ &= \left[C_1 \sin \omega^*(t-\tau) + C_2 \cos \omega^*(t-\tau)\right] J_0^* \end{aligned} \tag{7.38}$$

where ω^* is the frequency of the free vibrations, and λ^* determines the argument of the function J_0^*. The constants C_1 and C_2 are found from the initial conditions. Putting $t=\tau$ in (7.38), we obtain:

$$\left.\begin{aligned} w &= C_2 J_0^*, \\ \frac{\partial w}{\partial t} &= \omega^* C_1 J_0^*. \end{aligned}\right\} \tag{7.39}$$

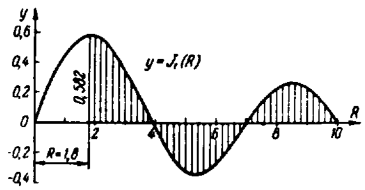


FIGURE 151.

Solving (7.37) and (7.39) for C_1 and C_2 yields:

$$\left.\begin{aligned} C_1 &= C\frac{\omega}{\omega^*}\cos\omega\tau\frac{J}{J_0^*}, \\ C_2 &= C\sin\omega\tau\frac{J_0}{J_0^*}, \end{aligned}\right\} \tag{7.40}$$

where C is given by (7.22) in the case of a flexible plate, and by (7.27) in the case of a massive plate.

Substitution of (7.40) in (7.38) yields:

$$w(\rho, t) = \left[\frac{\omega}{\omega^*}\cos\omega\tau\sin\omega^*(t-\tau) + \sin\omega\tau\cos\omega^*(t-\tau)\right] C J_0. \tag{7.41}$$

We obtain the parameter ω^* from (7.1), assuming its solution to be of the form:

$$\nabla^2 w = \alpha w, \tag{7.42}$$

where α has to be determined.

Substitution of (7.42) in (7.1) yields:

$$\alpha^2 w - 2r^2\alpha w + s^4 w = -m^*\frac{\partial^2 w}{\partial t^2}. \tag{7.43}$$

Inserting into this (7.41), we obtain:

$$\alpha^2 - 2r^2\alpha + s^4 = m^*\omega^{*2}, \tag{7.44}$$

where ω^* is to be considered as a function of α, whose extremum is given by:

$$\frac{d\omega^*}{d\alpha} = 0, \tag{7.45}$$

or

$$\frac{d\omega^*}{d\alpha} = \frac{\alpha - r^2}{m^* \sqrt{\frac{\alpha^2 - 2r^2\alpha + s^4}{m^*}}} = 0.$$

This condition is fulfilled for:

$$\alpha = r^2.$$

Substitution of this value of α in (7.44) yields:

$$\omega^* = \sqrt{\frac{s^4 - r^4}{m^*}}, \qquad (7.46)$$

where, as before:

$$m^* = \frac{m_1 + m_0}{D} = \left[\frac{\gamma h}{g} + \frac{\gamma_0}{g}\int_0^H \psi^2(z)\,dz\right]\frac{1}{D}.$$

§ 8. PLATE WITH SIMPLE SUPPORTS ALONG THE EDGES

2

Let the deflections of a freely vibrating plate with simple supports, resting on an elastic foundation (Figure 152) be given by:

$$w(x, y, t) = w(x, y)\,T(t). \qquad (8.1)$$

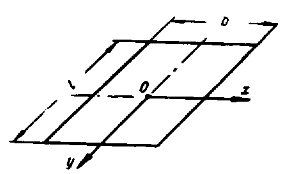


FIGURE 152.

The differential equation describing the shape of the deformed plate surface is, as before:

$$\nabla^2\nabla^2 w - 2r^2\nabla^2 w + (s^4 - m^*\omega^2)\,w = 0, \qquad (8.2)$$

whose solution can be represented as follows:

$$w(x, y) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} A_{in} \cos\frac{i\pi x}{b}\cos\frac{n\pi y}{l}. \qquad (8.3)$$

Each term of this series satisfies the boundary conditions at the plate edges [when i and n are odd]:

at $x = \pm\frac{b}{2}$: $\quad w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0;$

at $y = \pm\frac{l}{2}$: $\quad w = 0, \quad \frac{\partial^2 w}{\partial y^2} = 0.$

We determine the natural frequencies ω by substituting (8.3) in (8.2), multiplying the result by $\cos\frac{j\pi x}{b}\cos\frac{h\pi x}{l}$ (j and h are arbitrary integers), and integrating over the plate surface:

$$\int\limits_{-\frac{b}{2}}^{\frac{b}{2}}\int\limits_{-\frac{l}{2}}^{\frac{l}{2}}\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} A_{in}\left[\left(\frac{i^2\pi^2}{b^2}+\frac{n^2\pi^2}{l^2}\right)^2 + 2r^2\left(\frac{i^2\pi^2}{b^2}+\frac{n^2\pi^2}{l^2}\right) + s^4 - m^*\omega^2\right]\times$$
$$\times\cos\frac{i\pi x}{b}\cos\frac{n\pi y}{l}\cos\frac{j\pi x}{b}\cos\frac{h\pi y}{l}\,dx\,dy = 0. \tag{8.4}$$

Equation (8.4) defines the work done by all generalized forces acting on plate and elastic foundation over the virtual displacements of the plate. Because of the orthogonality of the trigonometric functions, all integrals for which $i \neq j$ and $n \neq h$ are equal to zero, and (8.4) reduces to:

$$A_{in}\iint\left[\left(\frac{i^2\pi^2}{b^2}+\frac{n^2\pi^2}{l^2}\right)^2 + 2r^2\left(\frac{i^2\pi^2}{b^2}+\frac{n^2\pi^2}{l^2}\right) + s^4 - m^*\omega^2\right]\times$$
$$\times\cos^2\frac{i\pi x}{b}\cos^2\frac{n\pi y}{l}\,dx\,dy = 0. \tag{8.5}$$

In order that the solution be nontrivial, we must have:

$$\left(\frac{i^2\pi^2}{b^2}+\frac{n^2\pi^2}{l^2}\right)^2 + 2r^2\left(\frac{i^2\pi^2}{b^2}+\frac{n^2\pi^2}{l^2}\right) + s^4 - m^*\omega^2 = 0,$$

whence:

$$\omega_{in} = \sqrt{\frac{1}{m^*}\left[\left(\frac{i^2\pi^2}{b^2}+\frac{n^2\pi^2}{l^2}\right)^2 + 2r^2\left(\frac{i^2\pi^2}{b^2}+\frac{n^2\pi^2}{l^2}\right) + s^4\right]}. \tag{8.6}$$

The entire frequency spectrum is obtained by assigning in turn different integral values to i and n. The principal vibration modes corresponding to these frequencies are:

$$w_{in} = A_{in}\cos\frac{i\pi x}{b}\cos\frac{n\pi y}{l}. \tag{8.7}$$

2

The coefficients A_{in} are determined from the initial conditions of the problem. Let a momentary impulse of intensity $p(x, y)$ per unit area act on the plate. Since there are no displacements at $t = 0$, the deflection function can be written:

$$w(x, y, t) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} A_{in} \cos \frac{i\pi x}{b} \cos \frac{n\pi y}{l} \sin \omega_{in} t, \tag{8.8}$$

whence:

$$\frac{\partial w}{\partial t} = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} A_{in}\, \omega_{in} \cos \frac{i\pi x}{b} \cos \frac{n\pi y}{l} \cos \omega_{in} t. \tag{8.9}$$

The initial velocity at $t = 0$ is:

$$\left(\frac{\partial w}{\partial t}\right)_{t=0} = \sum_{i=1}^{\infty} \sum^{\infty} A_{in}\, \omega_{in} \cos \frac{i\pi x}{b} \cos \frac{n\pi y}{l} = \frac{p(x, y)}{m}. \tag{8.10}$$

We multiply (8.10) by $\cos \frac{j\pi x}{b} \cos \frac{h\pi y}{l}$ and integrate the result over the plate surface. Because of the orthogonality of the trigonometric functions this yields:

$$A_{in}\, \omega_{in} \int_{-b/2}^{b/2} \int_{-l/2}^{l/2} \cos^2 \frac{i\pi x}{b} \cos^2 \frac{n\pi y}{l}\, dx\, dy = \frac{1}{m} \int_{-b/2}^{b/2} \int_{-l/2}^{l/2} p(x, y) \cos \frac{i\pi x}{b} \cos \frac{n\pi y}{l}\, dx\, dy,$$

whence:

$$A_{in} = \frac{4 \int_{-b/2}^{b/2} \int_{-l/2}^{l/2} p(x, y) \cos \frac{i\pi x}{b} \cos \frac{n\pi y}{l}\, dx\, dy}{m\, \omega_{in}\, lb} \tag{8.11}$$

Thus, finally:

$$w(x, y, t) = \frac{4}{m\, lb} \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \frac{\int_{-b/2}^{b/2} \int_{-l/2}^{l/2} p(x, y) \cos \frac{i\pi x}{b} \cos \frac{n\pi y}{l}\, dx\, dy}{\omega_{in}} \times \cos \frac{i\pi x}{b} \cos \frac{n\pi y}{l} \sin \omega_{in} t. \tag{8.12}$$

Substituting (8.12) in (1.8) of Chapter III, we obtain the values of the bending moments and shearing forces appearing in the plate due to the momentary impulse $p(x, y)$.

§ 9. VIBRATIONS OF A PLATE WITH FREE EDGES

1

The natural frequencies of a plate resting freely on an elastic foundation

(Figure 153) are determined by separating the variables in (6.10):

$$T'' + \omega^2 T = 0, \tag{9.1}$$

$$D\nabla^2\nabla^2 w - 2t\nabla^2 w + (k - m\omega^2)\, w = 0. \tag{9.2}$$

The shape of the deformed surface of the plate can be represented in the following form:

$$w(x, y) = \sum_1^m \sum_1^n C_{mn}\, \varphi_{mn}(x, y), \tag{9.3}$$

where φ_{mn} (x, y) = linearly independent functions, selected in advance according to the geometrical boundary conditions, and C_{mn} = constant coefficients. For the functions φ_{mn} we shall choose trigonometric functions together with linear terms corresponding to rigid-body displacements of the plate:

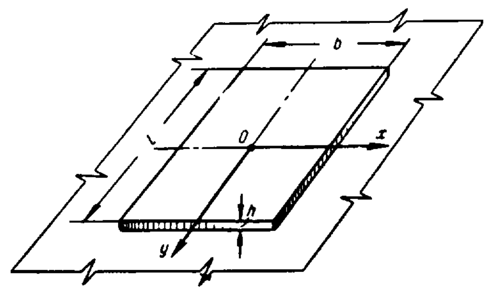


FIGURE 153.

The frequencies corresponding to the vibrational modes (9.3) are determined from the equilibrium conditions, applying Lagrange's principle of virtual displacements. For this we calculate the work done by all external and internal forces acting on the plate over any virtual displacement, in the same way as in section 12 of Chapter III. Substitution of (9.3) in (9.2) then yields the following system of algebraic equations:

$$\sum_1^m \sum_1^n C_{mn} \Big\{ \iint [D\nabla^2\nabla^2 \varphi_{mn} - 2t\nabla^2\varphi_{mn} + (k - m\omega^2)\, \varphi_{mn}]\, \varphi_{ik}\, dx\, dy + \\ + \oint [Q_{mn}(s) + Q^{\Phi}(s)]\, \varphi_{ik}\, ds \Big\} = 0 \tag{9.4}$$
$$(i = 1, 2, 3, \ldots, m; \qquad k = 1, 2, 3, \ldots, n),$$

The terms under the double integral sign in (9.4) represent the work done by the internal forces acting in the plate, the reactions of the elastic foundation, and the inertia forces arising in the plate and the elastic foundation. The contour integral represents the work done by the shearing forces acting on the plate edges. The first term determines the work done by Kirchhoff's reduced additional shearing forces (cf. (1.9) of Chapter III), which appear at the plate edges as a result of the approximate fulfilment of the static-equilibrium conditions by the functions $\varphi(x, y)$. The second term represents the work done by the reactions, distributed over the plate edges, determined by the deformations of the elastic foundation beyond the plate edges. Using (10.8) and (10.9) of Chapter III, and taking into account the work done by the inertia forces acting on the elastic foundation beyond the plate edges (cf. for instance (5.5) and (5.14)), we obtain:

$$\left.\begin{aligned} Q_l{}^{\Phi} &= 2t\left[\left(\alpha - m_0\frac{\omega^2}{4\alpha t}\right)w_l + \left(\frac{\partial w}{\partial x}\right)_l - \frac{1}{2\alpha}\left(\frac{\partial^2 w}{\partial y^2}\right)_l\right], \\ Q_b{}^{\Phi} &= 2t\left[\left(\alpha - m_0\frac{\omega^2}{4\alpha t}\right)w_b + \left(\frac{\partial w}{\partial y}\right)_b - \frac{1}{2\alpha}\left(\frac{\partial^2 w}{\partial x^2}\right)_b\right], \end{aligned}\right\} \tag{9.5}$$

where the subscripts l and b correspond to the longitudinal $(x=\pm b)$ and lateral $(y=\pm l)$ edges respectively.

Equations (9.4) holds true even when the postulation of a foundation modulus is acceptable; in this case the terms containing Q^{Φ} and t should be discarded, the characteristic k being taken as foundation modulus.

Equations (9.4) can be represented in the following form:

$$\left.\begin{aligned} \delta_{00,00}C_{00} + \delta_{00,10}C_{10} + \ldots + \delta_{00,mn}C_{mn} &= 0, \\ \delta_{10,00}C_{00} + \delta_{10,10}C_{10} + \ldots + \delta_{10,mn}C_{mn} &= 0. \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \delta_{ik,00}C_{00} + \delta_{ik,10}C_{10} + \ldots + \delta_{ik,mn}C_{mn} &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \delta_{mn,00}C_{00} + \delta_{mn,10}C_{10} + \ldots + \delta_{mn,mn}C_{mn} &= 0, \end{aligned}\right\} \tag{9.6}$$

where:

$$\begin{aligned} \delta_{ik,mn} = \iint [D\nabla^2\nabla^2\varphi_{mn} - 2t\nabla^2\varphi_{mn} + (k - m\omega^2)\varphi_{mn}]\,\varphi_{ik}\,dx\,dy + \\ + \oint [Q_{mn}(s) + Q^{\Phi}(s)]\,\varphi_{ik}(s)\,ds. \end{aligned} \tag{9.7}$$

Integration in (9.7) is extended over the entire surface and the entire contour of the plate respectively. These integrals represent the virtual work done by the forces corresponding to one state of the system over the displacements corresponding to another state. Hence, by the reciprocity theorem:

$$(\delta_{ik,mn} = \delta_{mn,ik})$$

and the matrix of (9.6) will be symmetrical.

System (9.6) will have a nontrivial solution if its determinant vanishes:

$$\begin{vmatrix} \delta_{00,00} & \delta_{00\,10} & \ldots & \delta_{00,mn} \\ \delta_{10,00} & \delta_{10,10} & \ldots & \delta_{10,mn} \\ \cdots & \cdots & \cdots & \cdots \\ \delta_{mn,00} & \delta_{mn,10} & \ldots & \delta_{mn,mn} \end{vmatrix} = 0. \tag{9.8}$$

Expansion of (9.8) leads to an equation of $(m+n)$-th degree in ω^2, whose solution will give all the $m+n$ frequencies corresponding to the vibrational modes (9.3). The natural vibrations of the plate are thus:

$$w(x, y, t) = \sum_{1}^{m} \sum_{1}^{n} C_{mn} \varphi_{mn}(x, y) \sin \omega_{mn} (t - \psi_{mn}), \tag{9.9}$$

where ψ_{mn} = constant determining the phase shift.

If the shape of the deformed surface of the plate is described by trigonometric functions, their orthogonality enables the general problem of the motion of a plate on an elastic foundation to be divided into four independent problems corresponding to the symmetrical and antisymmetrical vibrations relative to the x and y axes respectively. Each of these problems will now be treated separately.

2

In the case of symmetrical vibrations, (9.3) becomes:

$$\begin{gathered} w(x, y) = C_{00} + \sum_{1}^{m} C_{m0} \cos \frac{m\pi x}{b} + \sum_{1}^{n} C_{0n} \cos \frac{n\pi y}{l} + \\ + \sum_{1}^{m} \sum_{1}^{n} C_{mn} \cos \frac{m\pi x}{b} \cos \frac{n\pi y}{l} \\ (m \; n = 1, 3, 5, \ldots, (2k-1)). \end{gathered} \tag{9.10}$$

This means that:

$$\left. \begin{aligned} \varphi_{00} &= 1, \quad \varphi_{m0} = \cos \frac{m\pi x}{b}, \\ \varphi_{0n} &= \cos \frac{n\pi y}{l}, \quad \varphi_{mn} = \cos \frac{m\pi x}{b} \cos \frac{n\pi y}{l}, \end{aligned} \right\} \tag{9.11}$$

where m and n = odd integers.

Substitution of (9.11) in (9.7) yields all the coefficients in (9.6). For example, when four terms $(m = 1, n = 1)$, corresponding to the four possible modes of plate vibration shown, are taken in (9.10), the matrix of the algebraic equations is given in Table 18. This matrix is symmetrical. It is formed by calculating 10 dimensionless coefficients, using the following symbols:

$$\left. \begin{aligned} \alpha &= \sqrt{\frac{k}{2t}}, \qquad \beta = \frac{b}{l}, \\ D &= \frac{Eh^3}{12(1-\mu^2)}, \\ k &= \frac{E_0}{1-\nu_0^2} \int_0^H \psi'^2(z)\, dz, \\ t &= \frac{E_0}{4(1+\nu_0)} \int_0^H \psi^2(z)\, dz, \\ m_0 &= \bar{m}_0 \int_0^H \psi^2(z)\, dz, \end{aligned} \right| \tag{9.12}$$

Here l and b = length and width of the plate respectively, D = flexural rigidity of plate, m_0 = reduced mass of elastic foundation, k and t = generalized characteristics of elastic foundation, $\phi = \phi(t)$ = function representing distribution of displacements over thickness of elastic foundation.

By equating to zero the determinant of this matrix we obtain an algebraic equation of the fourth degree in ω^2, from which the natural frequencies of the plate, corresponding to the four assumed modes, can be obtained.

If the plate is considered to be perfectly rigid, all terms except the first in (9.10) vanish. From Table 18, we obtain for this case:

$$1+\frac{2}{ab}(1+\beta)-\frac{m_0}{k}\left(1+\frac{m_1}{m_0}+\frac{1+\beta}{ab}\right)\omega^2=0$$

or

$$\omega_{00}=\sqrt{\frac{k}{m_0}\,\frac{1+\frac{2}{ab}(1+\beta)}{1+\frac{m_1}{m_0}+\frac{1+\beta}{ab}}}. \qquad (9.13)$$

3

In the case of antisymmetrical vibrations, (9.3) becomes:

a) for vibrations symmetrical with respect to the x axis and antisym metrical with respect to the y axis:

$$w(x,y)=C_{00}\frac{2x}{b}+\sum_{2}^{m}C_{m0}\sin\frac{m\pi x}{b}+\frac{2x}{b}\sum_{1}^{n}C_{0n}\cos\frac{n\pi y}{l}+$$
$$+\sum_{2}^{m}\sum_{1}^{n}C_{mn}\sin\frac{m\pi x}{b}\cos\frac{n\pi y}{l} \qquad (9.14)$$
$$(m=2,4,6,\ldots;\ n=1,3,5,\ldots);$$

b) for vibrations symmetrical with respect to the y axis and antisym-metrical with respect to the x axis:

$$w(x,y)=C_{00}\frac{2y}{l}+\frac{2y}{l}\sum_{1}^{m}C_{m0}\cos\frac{m\pi x}{b}+\sum_{2}^{n}C_{0n}\sin\frac{n\pi y}{l}+$$
$$+\sum_{1}^{m}\sum_{2}^{n}C_{mn}\cos\frac{m\pi x}{b}\sin\frac{n\pi y}{l} \qquad (9.15)$$
$$(m=1,3,5,\ldots;\ n=2,4,6,\ldots);$$

c) for vibrations antisymmetrical with respect to both axes:

$$w(x,y)=C_{00}\frac{4xy}{lb}+\frac{2y}{l}\sum_{2}^{m}C_{m0}\sin\frac{m\pi x}{b}+$$
$$+\frac{2x}{b}\sum_{2}^{n}C_{0n}\sin\frac{n\pi y}{l}+\sum_{2}^{m}\sum_{2}^{n}C_{mn}\sin\frac{m\pi x}{b}\sin\frac{n\pi y}{l} \qquad (9.16)$$
$$(n,m=2,4,6,8,\ldots).$$

TABLE 18

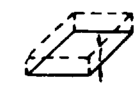
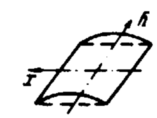
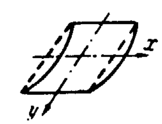
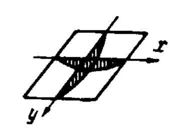
	C_{00}	C_{10}	C_{01}	C_{11}	Virtual displacements
0—0	$1+\frac{2}{\alpha b}(1+\beta)- -\frac{m_0}{k}\left(1+\frac{m_1}{m_0}+\frac{1+\beta}{\alpha\beta}\right)\omega^2$	$\frac{2}{\pi}\left[1+\frac{2}{\alpha b}\beta- -\frac{m_0}{k}\left(1+\frac{m_1}{m_0}+\frac{\beta}{\alpha b}\right)\omega^2\right]$	$\frac{2}{\pi}\left[1+\frac{2}{\alpha b}-\frac{m_0}{k}\times \times\left(1+\frac{m_1}{m_0}+\frac{1}{\alpha b}\right)\omega^2\right]$	$\frac{4}{\pi^2}\left(1-\frac{m_1+m_0}{k}\omega^2\right)$	$\bar{w}_{00}=1$ 
1—0	•	$\frac{1}{2}\left[\frac{\pi^4 D}{kb^4}+\frac{\pi^2}{\alpha^2 b^2}+1+\frac{2}{\alpha b}\beta+ +\frac{\pi^2}{\alpha^2 b^2}\beta-\frac{m_0}{k}\left(1+\frac{m_1}{m_0}+\frac{\beta}{\alpha\beta}\right)\omega^2\right]$	$\frac{4}{\pi^2}\left(1-\frac{m_1+m_0}{k}\omega^2\right)$	$\frac{1}{\pi}\left[\frac{\pi^4 D}{kb^4}+\frac{\pi^2}{\alpha^2 b^2}+1- -\frac{m_1+m_0}{k}\omega^2\right]$	$\bar{w}_{10}=\cos\frac{\pi x}{b}$ 
0—1	•	•	$\frac{1}{2}\left[\frac{\pi^4 D}{kb^4}\beta^4+\frac{\pi^2}{\alpha^2 b^2}\beta^2+ +1+\frac{2}{\alpha b}+\frac{\pi^2}{\alpha^2 b^2}\beta^2- -\frac{m_0}{k}\left(1+\frac{m_1}{m_0}+\frac{1}{\alpha b}\right)\omega^2\right]$	$\frac{1}{\pi}\left[\frac{\pi^4 D}{kb^4}\beta^4+\frac{\pi^2}{\alpha^2 b^2}\beta^2+ +1-\frac{m_1+m_0}{k}\omega^2\right]$	$\bar{w}_{01}=\cos\frac{\pi y}{l}$ 
1—1	•	•	•	$\frac{1}{4}\left[(1+\beta^2)^2\frac{\pi^4 D}{kb^4}+ +(1+\beta^2)\frac{\pi^2}{\alpha^2 b^2}+1- -\frac{m_1+m_0}{k}\omega^2\right]$	$\bar{w}_{11}=\cos\frac{\pi x}{b}\cos\frac{\pi y}{l}$ 

TABLE 19

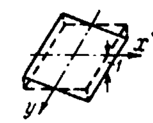
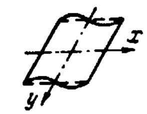
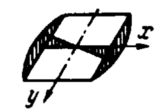
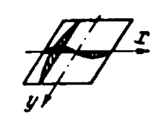
	C_{00}	C_{20}	C_{01}	C_{21}	Virtual displacements
0—0	$\frac{1}{3}\left[1+\frac{6}{\alpha b}\left(1+\frac{\beta}{3}\right)+\frac{12}{\alpha^2 b^2}-\right.$ $\left.-\frac{m_0}{k}\left(1+\frac{m_1}{m_0}+\frac{3+\beta}{\alpha b}\right)\omega^2\right]$	$\frac{1}{\pi}\left[1+\frac{2}{\alpha b}\beta-\right.$ $\left.-\frac{m_0}{k}\left(1+\frac{m_1}{m_0}+\frac{\beta}{\alpha b}\right)\omega^2\right]$	$\frac{2}{3\pi}\left[1+\frac{6}{\alpha b}+\frac{12}{\alpha^2 b^2}-\right.$ $\left.-\frac{m_0}{k}\left(1+\frac{m_1}{m_0}+\frac{3}{\alpha b}\right)\omega^2\right]$	$\frac{2}{\pi^2}\left(1-\frac{m_1+m_0}{k}\omega^2\right)$	$\bar{w}_{00}=\frac{2x}{b}$ 
2—0	•	$\frac{1}{2}\left[16\frac{\pi^4 D}{kb^4}+4\frac{\pi^2}{\alpha^2 b^2}+1+\frac{2}{\alpha b}\beta+\right.$ $\left.+4\frac{\pi^2}{\alpha^2 b^2}\beta-\frac{m_0}{k}\left(1+\frac{m_1}{m_0}+\frac{\beta}{\alpha b}\right)\omega^2\right]$	$\frac{2}{\pi^2}\left(1-\frac{m_1+m_0}{k}\omega^2\right)$	$\frac{1}{\pi}\left[16\frac{\pi^4 D}{kb^4}+4\frac{\pi^2}{\alpha^2 b^2}+1-\right.$ $\left.-\frac{m_1+m_0}{k}\omega^2\right]$	$\bar{w}_{20}=\sin\frac{2\pi x}{b}$ 
0—1	•	•	$\frac{1}{8}\left[\frac{\pi^4 D}{kb^4}\beta^4+\frac{\pi^2}{\alpha^2 b^2}\beta^2+1+\right.$ $+12(2-\mu)\frac{\pi^2 D}{kb^4}\beta^2+$ $+\frac{6}{\alpha b}+\frac{12}{\alpha^2 b^2}+$ $+3\frac{\pi^2}{\alpha^2 b^2}\beta^2-\frac{m_0}{k}\times$ $\left.\times\left(1+\frac{m_1}{m_0}+\frac{3}{\alpha b}\right)\omega^2\right]$	$\frac{1}{2\pi}\left[\frac{\pi^4 D}{kb^4}\beta^4+\frac{\pi^2}{\alpha^2 b^2}\beta^2+\right.$ $\left.+1-\frac{m_1+m_0}{k}\omega^2\right]$	$\bar{w}_{01}=\frac{2x}{b}\cos\frac{\pi y}{l}$ 
2—1	•	•	•	$\frac{1}{4}\left[(4+\beta^2)^2\frac{\pi^4 D}{kb}+\right.$ $+(4+\beta^2)\frac{\pi^2}{\alpha^2 b^2}+1-$ $\left.-\frac{m_1+m_0}{k}\omega^2\right]$	$\bar{w}_{21}=\sin\frac{2\pi x}{b}\cos\frac{\pi y}{l}$ 

TABLE 20

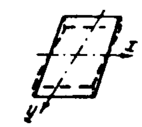
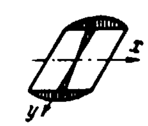
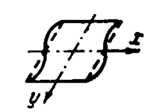
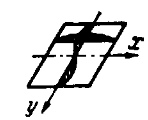
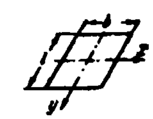
	C_{00}	C_{10}	C_{02}	C_{12}	Virtual displacements
0—0	$\frac{1}{3}\left[1+\frac{6}{\alpha b}\left(\beta+\frac{1}{3}\right)+\frac{12}{\alpha^2 b^2}\beta^2-\right.$ $\left.-\frac{m_0}{k}\left(1+\frac{m_1}{m_0}+\frac{3\beta+1}{\alpha b}\right)\omega^2\right]$	$\frac{2}{3\pi}\left[1+\frac{6}{\alpha b}\beta+\frac{12}{\alpha^2\beta^2}\beta^2-\right.$ $\left.-\frac{m_0}{k}\left(1+\frac{m_1}{m_0}+\frac{3\beta}{\alpha b}\right)\omega^2\right]$	$\frac{1}{\pi}\left[1+\frac{2}{\alpha b}-\frac{m_0}{k}\times\right.$ $\left.\times\left(1+\frac{m_1}{m_0}+\frac{1}{\alpha b}\right)\omega^2\right]$	$\frac{2}{\pi^2}\left(1-\frac{m_1+m_0}{k}\omega^2\right)$	$\bar{w}_{00}=\frac{2y}{l}$ 
1—0	•	$\frac{1}{6}\left[\frac{\pi^4 D}{kb^4}+\frac{\pi^2}{\alpha^2 b^2}+1+\right.$ $+12(2-\mu)\frac{\pi^2 D}{kb^4}\beta^2+\frac{6}{\alpha b}\beta+$ $+\frac{12}{\alpha^2 b^2}\beta^2+3\frac{\pi^2}{\alpha^2 b^2}\beta-$ $\left.-\frac{m_0}{k}\left(1+\frac{m_1}{m_0}+\frac{3\beta}{\alpha b}\right)\omega^2\right]$	$\frac{2}{\pi^2}\left(1-\frac{m_1+m_0}{k}\omega^2\right)$	$\frac{1}{2\pi}\left[\frac{\pi^4 D}{kb^4}+\frac{\pi^2}{\alpha^2 b^2}+1-\right.$ $\left.-\frac{m_1+m_0}{k}\omega^2\right]$	$\bar{w}_{10}=\frac{2y}{l}\cos\frac{\pi x}{b}$ 
0—2	•	•	$\frac{1}{2}\left[16\frac{\pi^4 D}{kb^4}\beta^4+\right.$ $+4\frac{\pi^2}{\alpha^2 b^2}\beta^2+1+$ $+\frac{2}{\alpha b}+4\frac{\pi^2}{\alpha^2 b^2}\beta^2-$ $\left.-\frac{m_0}{k}\left(1+\frac{m_1}{m_0}+\frac{1}{\alpha b}\right)\omega^2\right]$	$\frac{1}{\pi}\left[16\frac{\pi^4 D}{kb^4}\beta^4+\right.$ $+4\frac{\pi^2}{\alpha^2 b^2}\beta^2+1-$ $\left.-\frac{m_1+m_0}{b}\omega^2\right]$	$\bar{w}_{02}=\sin\frac{2\pi y}{l}$ 
1—2	•	•	•	$\frac{1}{4}\left[(1+4\beta^2)^2\frac{\pi^4 D}{kb^4}+\right.$ $+(1+4\beta^2)\frac{\pi^2}{\alpha^2 b^2}+1-$ $\left.-\frac{m_1+m_0}{k}\omega^2\right]$	$\bar{w}_{12}=\cos\frac{\pi x}{b}\sin\frac{2\pi y}{l}$ 

TABLE 21

	C_{00}	C_{20}	C_{02}	C_{22}	Virtual displacements
0—0	$\frac{1}{9}\Big[288(1-\mu)\frac{D}{kb^4}\beta^2+1+\frac{6}{\alpha\beta}(1+\beta)+\frac{12}{\alpha^2b^2}(1+\beta^2)-\frac{m_0}{k}\Big(1+\frac{m_1}{m_0}+\frac{3(1+\beta)}{\alpha b}\Big)\omega^2\Big]$	$\frac{1}{3\pi}\Big[1+\frac{6}{\alpha b}\beta+\frac{12}{\alpha^2b^2}\beta^2-\frac{m_0}{k}\Big(1+\frac{m_1}{m_0}+\frac{3\beta}{\alpha\beta}\Big)\omega^2\Big]$	$\frac{1}{3\pi}\Big[1+\frac{6}{\alpha b}+\frac{12}{\alpha^2b^2}-\frac{m_0}{k}\Big(1+\frac{m_1}{m_0}+\frac{3}{\alpha b}\Big)\omega^2\Big]$	$\frac{1}{\pi^2}\Big(1-\frac{m_1+m_0}{k}\omega^2\Big)$	$\bar{w}_{00}=\frac{4xy}{lb}$ 
2—0	•	$\frac{1}{6}\Big[16\frac{\pi^4D}{kb^4}+4\frac{\pi^2}{\alpha^2b^2}+1+48(2-\mu)\frac{\pi^2D}{kb^4}\beta^2+\frac{6}{\alpha b}\beta+\frac{12}{\alpha^2b^2}\beta^2+12\frac{\pi^2}{\alpha^2b^2}\beta-\frac{m_0}{k}\Big(1+\frac{m_1}{m_0}+\frac{3\beta}{\alpha b}\Big)\omega^2\Big]$	$\frac{1}{\pi^2}\Big(1-\frac{m_1+m_0}{k}\omega^2\Big)$	$\frac{1}{2\pi}\Big(16\frac{\pi^4D}{kb^4}+4\frac{\pi^2}{\alpha^2b^2}+1-\frac{m_1+m_0}{k}\omega^2\Big)$	$\bar{w}_{20}=\frac{2y}{l}\sin\frac{2\pi x}{b}$
0—2	•	•	$\frac{1}{6}\Big[16\frac{\pi^4D}{kb^4}\beta^4+4\frac{\pi^2}{\alpha^2b^2}\beta^2+1+48(2-\mu)\times\frac{\pi^2D}{kb^4}\beta^2+\frac{6}{\alpha b}+\frac{12}{\alpha^2b^2}+12\frac{\pi^2}{\alpha^2b^2}\beta^2-\frac{m_0}{k}\Big(1+\frac{m_1}{m_0}+\frac{3}{\alpha b}\Big)\omega^2\Big]$	$\frac{1}{2\pi}\Big(16\frac{\pi^4D}{kb^4}\beta^4+4\frac{\pi^2}{\alpha^2b^2}+1-\frac{m_1+m_0}{k}\omega^2\Big)$	$\bar{w}_{02}=\frac{2x}{b}\sin\frac{2\pi y}{l}$
2—2	•	•		$\frac{1}{4}\Big[16(1+\beta^2)^2\frac{\pi^4D}{kb^4}+4(1+\beta^2)\frac{\pi^2}{\alpha^2b^2}+1-\frac{m_1+m_0}{k}\omega^2\Big]$	$\bar{w}_{22}=\sin\frac{2\pi x}{b}\sin\frac{2\pi y}{l}$

The coefficients in (9.6) are again obtained from (9.7).

Considering only the first four terms in (9.14), (9.15), and (9.16), we can represent (9.6) for the cases a, b, and c, by the matrices given in Tables 19, 20, and 21 respectively, where α, β, k, D, and m_0, are given, as before, by (9.12). Equating to zero the determinants of these matrices, we obtain equations of the fourth degree in ω^2 which yield four natural frequencies for each case considered.

Taking only the first (linear) terms in (9.14) and (9.15), we find from Tables 19 and 20 the frequencies of a rigid plate:

in the case of vibrations antisymmetrical with respect to the y axis

$$\omega_{00}=\sqrt{\frac{k}{m_0}\,\frac{1+\frac{6}{ab}\left(1+\frac{\beta}{3}\right)+\frac{12}{a^2b^2}}{1+\frac{m_1}{m_0}+\frac{3+\beta}{ab}}}\,; \tag{9.17}$$

in the case of vibrations antisymmetrical with respect to the x axis

$$\omega_{00}=\sqrt{\frac{k}{m_0}\,\frac{1+\frac{6}{ab}\left(\frac{1}{3}+\beta\right)+\frac{12}{a^2b^2}\beta^2}{1+\frac{m_1}{m_0}+\frac{1+3\beta}{ab}}}\,. \tag{9.18}$$

The first approximation in the case of vibrations antisymmetrical with respect to both axes (Table 21) is:

$$\omega_{00}=\sqrt{\frac{k}{m_0}\,\frac{288\,(1-\mu)\frac{D}{kb^4}\beta^2+1+\frac{6}{ab}(1+\beta)+\frac{12}{a^2b^2}(1+\beta^2)}{1+\frac{m_1}{m_0}+\frac{3+3\beta}{ab}}} \tag{9.19}$$

§ 10. BUCKLING OF A RECTANGULAR PLATE RESTING ON AN ELASTIC SINGLE-LAYER FOUNDATION AND COMPRESSED IN ONE DIRECTION

Consider a rectangular plate resting on an elastic single-layer foundation and loaded by axial compressive forces $N(x)$ per unit width (Figure 154).

The differential equation of the deflections of this plate is:

$$D\nabla^2\nabla^2 w-2t\nabla^2 w+kw=-N(x)\frac{\partial^2 w}{\partial y^2}, \tag{10.1}$$

where D = flexural rigidity of the plate, and k and t = generalized characteristics of the elastic foundation. The compressive forces are considered positive in (10.1).

We represent the plate deflections by the finite series

$$w(x,y)=\sum_{k=1}^{n} W_k(y)\,\chi_k(x) \tag{10.2}$$

and assume as before (cf. section 2 of Chapter III) that the functions $\chi_k(x)$ are known; the functions $W_k(y)$ are considered as unknowns. We can then write (10.1) as follows: [cf. (2.25) of Chapter III]:

$$\sum_{k=1}^{n} a_{ik} W_k^{\mathrm{IV}} - \sum_{k=1}^{n} [2(b_{ik} + \rho_{ik}^0) - N_{ik}] W_k'' + \sum_{k=1}^{n} (c_{ik} + s_{ik}^0) W_k = 0 \qquad (i = 1, 2, 3, \ldots, n). \tag{10.3}$$

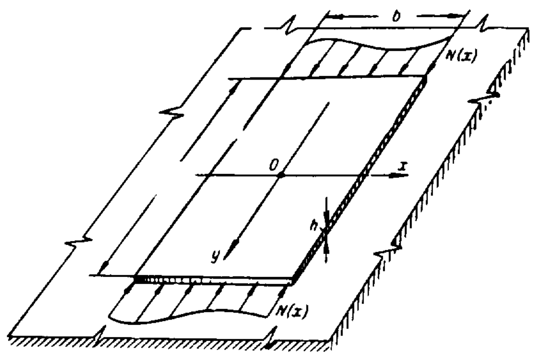


FIGURE 154.

The coefficients a_{ik}, b_{ik}, c_{ik}, ρ_{ik}^0, s_{ik}^0 in (10.3) depend on the selected system of functions $\chi_k(x)$ and on the values of the elastic constants of plate and foundation: [cf. (2.26) of Chapter III]:

$$\left.\begin{aligned}
a_{ik} &= \Sigma D \int \chi_k \chi_i \, dx, \\
b_{ik} &= \Sigma D \left\{ \int \chi_k' \chi_i' \, dx - \frac{\mu}{2} [\chi_k \chi_i' + \chi_k' \chi_i|^* \right\}, \\
c_{ik} &= \Sigma D \int \chi_k'' \chi_i'' \, dx, \\
\rho_{ik}^0 &= t \int \chi_k \chi_i \, dx + \frac{t}{2a} [[\chi_k \chi_i]], \\
s_{ik}^0 &= k \left\{ \int \chi_k \chi_i \, dx + \frac{2t}{k} \int \chi_k' \chi_i' dx + \frac{2at}{k} [[\chi_k \chi_i]] \right\}
\end{aligned}\right\} \tag{10.4}$$

The magnitudes N_{ik}, which depend on the compressive load $N(x)$, are:

$$N_{ik} = \int N(x) \chi_i \chi_k \, dx. \tag{10.5}$$

If the external load is uniformly distributed, i.e., if $N(x) = \text{const}$, equation (10.3) can be written in the form:

$$\sum_{k=1}^{n} a_{ik} W_k^{\mathrm{IV}} - \sum_{k=1}^{n} \left[2(b_{ik} + \rho_{ik}^0) - \frac{P}{bD} a_{ik} \right] W_k'' + \sum_{k=1}^{n} (c_{ik} + s_{ik}^0) W_k = 0, \tag{10.6}$$

where P = total compressive load and b = plate width.

By assigning to i successively all values from 1 to n we obtain from (10.6) or (10.3) a complete system of ordinary homogeneous differential equations in the unknown functions $W_k(y)\ (k=1,2,\ldots,n)$. With the functions $\chi_k(x)\ (k=1,2,\ldots,n)$ known, these equations can be solved up to a parameter P (N in the general case), representing the unknown critical force. By adding to (10.6) the homogeneous boundary conditions at $y = \pm l/_2$ for the functions $W_k(y)$, we obtain, from the conditions of the existence of nontrivial solutions, an infinite set of values for P. Since the system (10.6) has a symmetrical structure, the eigenvalues will always be real in the homogeneous boundary-value problem considered.

The best way to solve ordinary differential equations with constant coefficients is Krylov's method, which was developed for the case of small vibrations of systems with many degrees of freedom.

If $l \gg b$, the solution of (10.6) can be represented in the form:

$$W_k(y) = C_k \sin\frac{\pi y}{\lambda} \quad (k = 1, 2, 3, \ldots, n), \tag{10.7}$$

where the C_k = constants, and λ = length of the sine half-wave corresponding to buckling in the y direction.

Substituting (10.7) in (10.6) and equating to zero the determinant of the equations obtained (the C_k being considered as unknowns), we obtain a characteristic equation of order n in P, whose roots will be real. Since two unknowns, the force P and the half-wave length λ, are interrelated by the characteristic equation for finite values of n, these unknowns have to be found from condition:

$$\frac{dP}{d\lambda} = 0.$$

In practice it is sufficient to take only the first terms of (10.3). Buckling in the direction of the plate width is in this case characterized by the function $\chi(x)$, and the differential equation becomes:

$$AW^{\mathrm{IV}} + (N_{11} - 2B)W'' + CW = 0, \tag{10.8}$$

where $W = W(y)$ = unknown generalized deflection; and:

$$\left.\begin{aligned} A &= D\int \chi^2\,dx,\\ B &= D\left\{\int \chi'^2\,dx - \mu\,[\chi\chi']^*\right\} + t\int \chi^2\,dt + \frac{t}{2a}[[\chi^2]],\\ C &= D\int \chi''^2\,dx + k\left\{\int \chi^2\,dx + \frac{2t}{k}\int \chi'^2\,dx + \frac{2at}{k}[[\chi^2]]\right\},\\ N_{11} &= \int N(x)\,\chi^2\,dx. \end{aligned}\right\} \tag{10.9}$$

For $(l \ll b)$ we can represent the solution of (10.8) in the form (10.7). We obtain:

$$A\left(\frac{\pi}{\lambda}\right)^4 - (N_{11} - 2B)\left(\frac{\pi}{\lambda}\right)^2 + C = 0. \tag{10.10}$$

It can be seen from (10.10) that the generalized compressive force N_{11} is a function of λ. To determine the minimum (critical) value of n_{11}, we differentiate (10.10) with respect to λ and equate the result to zero. We obtain:

$$\lambda = \pi \sqrt[4]{\frac{A}{C}}. \tag{10.11}$$

The critical value of N_{11} is then:

$$N_{11} = 2(B + \sqrt{AC}). \tag{10.12}$$

If $N(x) = \text{const}$, we can rewrite (10.12) as follows:

$$P = \frac{2Db}{A}(B + \sqrt{AC}), \tag{10.13}$$

where P = total compressive load.

Introducing the generalized geometrical characteristics:

$$r^2 = \frac{Bl^2}{A}, \quad s^4 = \frac{Cl^4}{A}, \tag{10.14}$$

we can represent (10.11) and (10.13) in the following form:

$$\lambda = \frac{\pi}{s} l, \qquad P = \frac{EF}{6(1-\mu^2)} \frac{h^2}{l^2}(r^2 + s^2), \tag{10.15}$$

where F = area of plate cross section, h = plate thickness, and l = plate length. If the plate length l is of the order of the width b, but less than the wave length λ obtained from (10.11), then l has to be taken as wave length, and the critical compressive force is then determined from (10.10). If $\frac{l}{n} > \lambda > \frac{l}{n+1}$, we put $\lambda = \frac{l}{n}$ and $\frac{l}{n+1}$ in (10.10); the lower value obtained for N_{11} is the critical one.

§ 11. BUCKLING OF A NARROW PLATE RESTING ON AN ELASTIC SINGLE-LAYER FOUNDATION

Equations (10.3) are easiest to solve when the plate cross section can be considered as undeformable. This assumption is justified for sufficiently long plates with free edges (Figure 154), or with one edge simply supported (Figure 156), and also in many other cases when an elementary transverse strip of width dy, cut out from the plate, can be deformed (Figure 155). In this case we choose as functions $\chi_k(x)$ the displacements of the strip considered as a combination of rigid links.

We shall now consider some examples.

1. Rectangular plate simply supported along a longitudinal edge

Consider a rectangular plate of uniform thickness h, loaded by a centrally applied compressive force $P = Nb$ (Figure 156). Rigid-body rotation of the plate about the supported edge is taken as the virtual displacement depending on the x coordinate:

$$\chi = x.$$

This problem is described by (10.8), where

$$\left.\begin{aligned} A &= D\frac{b^3}{3},\\ B &= Db(1-\mu) + \frac{tb^3}{3}\left(1 + \frac{3}{2ab}\right),\\ C &= \frac{kb^3}{3}\left[1 + \frac{3}{ab}\left(1 + \frac{1}{ab}\right)\right],\\ N_{11} &= \frac{Pb^2}{3},\end{aligned}\right\} \qquad (11.1)$$

so that (10.8) becomes:

$$W^{IV} + \left[\frac{P}{Db} - \frac{6}{b^2}(1-\mu) - \frac{2t}{D}\left(1 + \frac{3}{ab}\right)\right]W'' + \frac{k}{D}\left[1 + \frac{3}{ab}\left(1 + \frac{1}{ab}\right)\right]W = 0. \qquad (11.2)$$

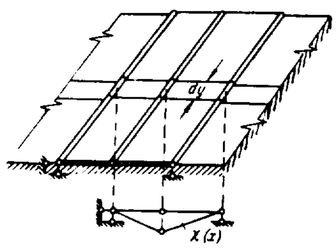


FIGURE 155.

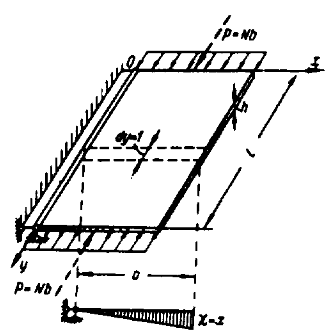


FIGURE 156.

Let the lateral plate edges $y = 0$ and $y = l$ be simply supported. The solution of (11.1) can then be represented in the form:

$$W = C\sin\frac{\pi y}{l}. \qquad (11.3)$$

Plates having undeformable cross sections can only buckle in the form of one half-wave, so that λ in (10.7) is always equal to the plate length l.

Substituting (11.3) in (11.2) and dividing by $C\sin\frac{\pi y}{l}$, which is different from zero, we obtain:

$$\frac{\pi^4}{l^4} - \left[\frac{P}{Db} - \frac{6}{b^2}(1-\mu) - \frac{2t}{D}\left(1 + \frac{3}{ab}\right)\right]\frac{\pi^2}{l^2} + \frac{k}{D}\left[1 + \frac{3}{ab}\left(1 + \frac{1}{ab}\right)\right] = 0,$$

whence:

$$P_{cr} = \frac{\pi^2 EJ}{l^2(1-\mu^2)} + \frac{6EJ}{b^2(1+\mu)} + 2tb\left(1 + \frac{3}{ab}\right) + \frac{kl^2b}{\pi^2}\left[1 + \frac{3}{ab}\left(1 + \frac{1}{ab}\right)\right]. \qquad (11.4)$$

where $J = \frac{bh^3}{12}$ = moment of inertia of the plate cross section.

For $\mu = 0$, the first term of (11.4) is identical with the expression for the critical Euler load. The second term is due to the increase in the critical force, caused by the fastening of the longitudinal edge of the plate. The last terms of (11.4) takes into account the supporting effect of the elastic foundation.

If the plate lies on a foundation for which a foundation modulus can be postulated, the terms containing t and α in (11.4) (determined by the shearing strains of the elastic foundation) should be discarded. We then obtain:

$$P_{cr} = \frac{\pi^2 EJ}{l^2(1-\mu)} + \frac{6EJ}{b^2(1+\mu)} + \frac{kbl^2}{\pi^2}, \tag{11.5}$$

where k = foundation modulus.

2. Rectangular plate with free longitudinal edges

Consider a rectangular plate loaded by axial forces whose transverse distribution is linear (Figure 157). These forces can be reduced to a centrally applied compressive force P (considered positive) and a bending moment M acting in the plane of the plate. We assumed that the plate consists of longitudinal strips of different thickness and that in the general case it has no longitudinal axis of symmetry.

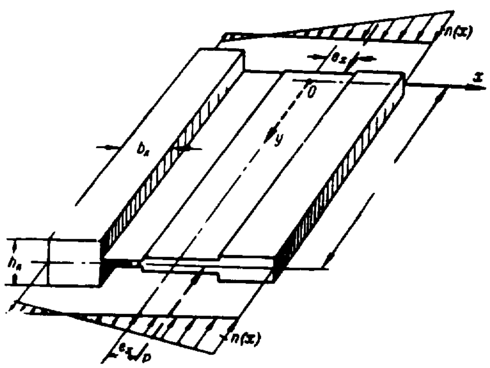


FIGURE 157.

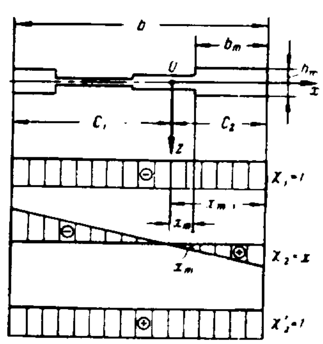


FIGURE 158.

The virtual displacements of a transverse strip, cut out from the plate, are taken as the translatory displacement $\chi_1 = 1$ and the rotation $\chi_2 = x$ about an axis passing through the centroid of the plate cross section (Figure 158).

The plate deflections are then:

$$w(x, y) = W_1\chi_1 + W_2\chi_2 = W_1 + W_2 x, \tag{11.6}$$

where W_1 and W_2 = generalized deflections. The functions W_1, which has the dimension of length, corresponds to the cylindrical bending of the plate in the longitudinal direction; the dimensionless function W_2 defines the angle of rotation about the axis through the centroid of the cross section. It is thus assumed that both flexural and torsional buckling of the plate is possible.

In the case considered (10.3) takes the form:

$$\left.\begin{aligned} &a_{11}W_1^{IV}+a_{12}W_2^{IV}+[N_{11}-2(b_{11}+\rho_{11}^0)]\,W_1^{\cdot\cdot}+[N_{12}-\\ &\quad-2(b_{12}+\rho_{12}^0)]\,W_2^{\cdot\cdot}+(c_{11}+s_{11}^0)\,W_1+(c_{12}+s_{12}^0)\,W_2=0,\\ &a_{21}W_1^{IV}+a_{22}W_2^{IV}+[N_{21}-2(b_{21}+\rho_{21}^0)]\,W_1^{\cdot\cdot}+[N_{22}-\\ &\quad-2(b_{22}+\rho_{22}^0)]\,W_2^{\cdot\cdot}+(c_{21}+s_{21}^0)\,W_1+(c_{22}+s_{22}^0)\,W_2=0,\end{aligned}\right\} \tag{11.7}$$

where by (10.4),

$$\left.\begin{aligned} &a_{11}=\sum D_m b_m, \qquad a_{12}=\sum D_m\frac{x_{m+1}^2-x_m^2}{2},\\ &a_{22}=\sum D_m\frac{1}{3}(x_{m+1}-x_m)(x_m^2+x_{m+1}x_m+x_{m+1}^2),\\ &b_{11}=b_{12}=0, \qquad b_{22}=\sum D_m b_m,\\ &c_{11}=c_{12}=c_{22}=0,\\ &\rho_{11}^0=tb\left(1+\frac{1}{ab}\right), \qquad \rho_{12}^0=\frac{t}{2}(c_2^2-c_1^2)\left(1+\frac{1}{ab}\right),\\ &\rho_{22}^0=tb\left[\frac{1}{3}(c_1^2+c_2^2-c_1c_2)+\frac{1}{2ab}(c_1^2+c_2^2)\right],\\ &s_{11}^0=kb\left(1+\frac{2}{ab}\right), \qquad s_{12}^0=\frac{k}{2}(c_2^2-c_1^2)\left(1+\frac{2}{ab}\right),\\ &s_{22}^0=kb\left[\frac{1}{3}(c_1^2+c_2^2-c_1c_2)+\frac{1}{ab}(c_1^2+c_2^2)+\frac{1}{a^2}\right].\end{aligned}\right\} \tag{11.8}$$

The summation in these expressions is extended over all the longitudinal strips.

Here $D=\frac{Eh^3}{12(1-\mu^2)}$ = flexural rigidity; b = overall width of the plate; k and t = compression and shear characteristics of elastic foundation respectively.

According to (10.5) we have:

$$\left.\begin{aligned} N_{11}&=\int N(x)\,\chi_1^2\,dx,\\ N_{12}&=\int N(x)\chi_1\chi_2\,dx,\\ N_{22}&=\int N(x)\,\chi_2^2\,dx.\end{aligned}\right\} \tag{11.9}$$

We can write:

$$N(x)=n(x)\,h, \tag{11.10}$$

where the normal stresses $N(x)$ are given by:

$$n(x)=\frac{P}{F}+\frac{M}{J_2}x. \tag{11.11}$$

Here F = the area, and $J_2 = \int x^2 dF$, the moment of inertia of the plate cross section; $M = Pe_x$, where e_x = eccentricity of applied force P.

Substituting (11.10) and (11.11) in (11.9), with $\chi_1 = 1$ and $\chi_2 = x$, we obtain:

$$\left.\begin{aligned} N_{11} &= \int nh\,dx = P, \\ N_{12} &= \int nxh\,dx = M, \\ N_{22} &= \int nx^2h\,dx = \frac{PJ_2}{F} + \frac{MJ_3}{J_2}, \end{aligned}\right\} \qquad (11.12)$$

where

$$J_3 = \int x^3\,dF, \quad dF = h\,dx. \qquad (11.13)$$

Substitution of (11.8) and (11.12) in (11.7) yields finally:

$$\left.\begin{aligned} a_{11}W_1^{IV} + (P - 2\rho_{11}^0)W_1'' + s_{11}^0W_1 + a_{12}W_2^{IV} + (M - 2\rho_{12}^0)\,W_2'' + \\ + s_{12}^0W_2 = 0, \\ a_{21}W_1^{IV} + (M - 2\rho_{21}^0)\,W_1'' + s_{21}^0W_1 + a_{22}W_2^{IV} + \\ + \left[\frac{PJ_2}{F} + \frac{MJ_3}{J_2} - 2\,(b_{22} + \rho_{22}^0)\right]W_2'' + s_{22}^0W_2 = 0. \end{aligned}\right\} \qquad (11.14)$$

Several examples will be given to illustrate the procedure adopted.

3. Rectangular plate of uniform section

If a rectangular plate of uniform thickness h has free longitudinal edges and is loaded by a centrally applied compressive force P (Figure 159), (11.8) and (11.12) reduce to:

$$\left.\begin{aligned} & a_{11} = Db = \frac{EJ}{1-\mu^2}, \quad a_{12} = b_{11} = b_{12} = c_{11} = c_{12} = c_{22} = 0, \\ & a_{22} = D\frac{b^3}{12} = \frac{EJb^2}{12\,(1-\mu^2)}, \quad b_{22} = Db = \frac{EJ}{1-\mu^2}, \\ & \rho_{11}^0 = tb\left(1 + \frac{1}{\alpha b}\right), \quad \rho_{12}^0 = s_{12}^0 = 0, \quad \rho_{22}^0 = \frac{tb^3}{12}\left(1 + \frac{3}{\alpha b}\right), \\ & s_{11}^0 = kb\left(1 + \frac{2}{\alpha b}\right), \quad s_{22}^0 = \frac{kb^3}{12}\left(1 + \frac{6}{\alpha b} + \frac{12}{\alpha^2 b^2}\right), \\ & N_{11} = P, \quad N_{12} = 0, \quad N_{22} = \frac{PJ_2}{F} = \frac{Pb^2}{12}. \end{aligned}\right\} \qquad (11.15)$$

System (11.14) can in this case be separated into two independent equations:

$$a_{11}W_1^{IV} + (P - 2\rho_{11}^0)\,W_1'' + s_{11}^0W_1 = 0, \qquad (11.16)$$

corresponding to flexural buckling,

and

$$a_{22}W_2^{IV} + \left[P\frac{b^2}{12} - 2\,(b_{22} + \rho_{22}^0)\right]W_2'' + s_{22}^0W_2 = 0, \qquad (11.17)$$

corresponding to torsional buckling.

Let the lateral plate edges be simply supported.

Substituting
$$W_1 = C_1 \sin\frac{\pi x}{l}, \quad W_2 = C_2 \sin\frac{\pi x}{l} \tag{11.18}$$
in (11.16) and (11.17), we obtain the following expressions for the critical forces:

$$P_1 = \frac{EJ\pi^2}{l^2(1-\mu^2)} + 2tb\left(1 + \frac{1}{ab}\right) + k\frac{bl^2}{\pi^2}\left(1 + \frac{2}{ab}\right), \tag{11.19}$$

$$P_2 = \frac{EJ\pi^2}{l^2(1-\mu^2)} + \frac{24EJ}{b^2(1-\mu^2)} + 2tb\left(1 + \frac{3}{ab}\right) + k\frac{bl^2}{\pi^2}\left(1 + \frac{6}{ab} + \frac{12}{a^2b^2}\right). \tag{11.20}$$

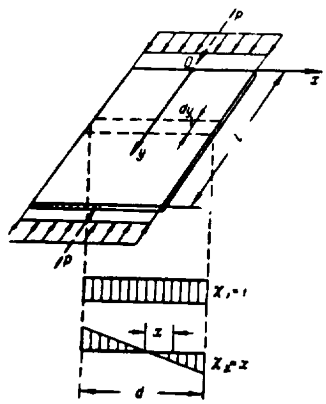


FIGURE 159.

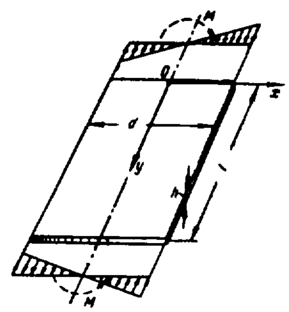


FIGURE 160.

It can be seen that in the case considered, the smaller critical force is given by (11.19), which corresponds to flexural buckling; in other words, torsional buckling is impossible in the symmetrical case. Let a moment M act on the same plate (Figure 160). The coefficients will have the values given by (11.15), except for N_{ik}. Here:

$$N_{11} = 0, \ N_{12} = M, \ N_{22} = M\frac{J_3}{J_2}. \tag{11.21}$$

For a symmetrical plate:

$$J_3 = \int x^3 \, dF = 0$$

so that $N_{22} = 0$. Hence, the system of equations (11.14) reduces to:

$$\left.\begin{aligned} a_{11}W_1^{IV} - 2p_{11}^0 W_1'' + s_{11}^0 W_1 + MW_2'' &= 0, \\ MW_1'' + a_{22}W_2^{IV} - 2(b_{22} + p_{22}^0)W_2'' + s_{22}^0 W_2 &= 0. \end{aligned}\right\} \tag{11.22}$$

When the lateral edges are simply supported, the solution of (11.22) can again be presented in the form (11.18).

Substitution of (11.18) in (11.22) yields the following expression for the critical moment:

$$M = \sqrt{\left(a_{11} + 2\rho_{11}^0 \frac{l^2}{\pi^2} + s_{11}^0 \frac{l^4}{\pi^4}\right)\left[a_{22} + 2\left(b_{22} + \rho_{22}^0\right)\frac{l^2}{\pi^2} + s_{22}^0 \frac{l^4}{\pi^4}\right]}. \qquad (11.23)$$

In this case, mixed flexural-torsional buckling takes places.

§ 12. BUCKLING OF A PRESTRESSED PLATE RESTING ON AN ELASTIC FOUNDATION

1

Consider a prestressed rectangular plate, compressed by a reinforcement rod lying in the longitudinal section $x = e_x$ (Figure 161). The rod is extended by a force $R = n_2 \Delta F$, where ΔF = cross-sectional area of bar.

The normal-stress diagram for the plate cross section $y = \text{const}$ is shown in Figure 161. With the exception of the vicinity of the rod, the normal-stress distribution is given by:

$$n_1 = \frac{R}{F} + \frac{Re_x}{J} x, \qquad (12.1)$$

where R = tensile force acting on the reinforcement rod.

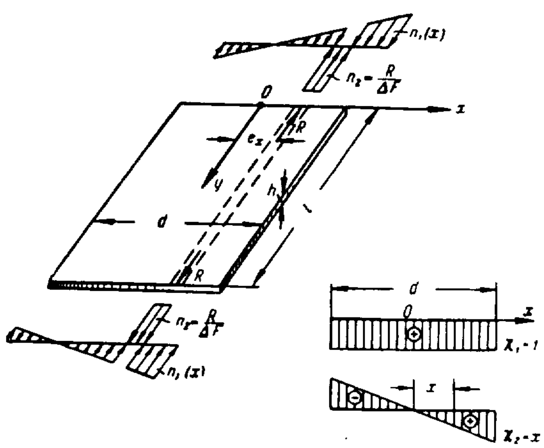


FIGURE 161.

The state of stress thus corresponds to a balanced (in static equilibrium) system of forces, i.e.:

$$\int_0^b n(x)\,dF = P = 0, \quad \int_0^b n(x)\,x\,dF = M = 0. \qquad (12.2)$$

If we assume that the plate cross section is not deformed, the solution will be given as before by (11.14). Those coefficients which do not depend on the compressive load are for a plate of uniform thickness determined by (11.15).

The coefficients which depend on the external load are obtained from (11.9), where the integrals are to be understood as Stieltjes integrals. The integration yields:

$$\left.\begin{aligned} N_{11} &= \int n(x)\,dF = \int n_1\,dF - n_2\,\Delta F = 0, \\ N_{12} &= \int n(x)\,x\,dF = \int n_1 x\,dF - n_2\,\Delta F e_x = 0, \\ N_{22} &= \int n(x)\,x^2\,dF = \frac{RJ_2}{F} + \frac{Re_x J_3}{J_2} - Re_x^2, \end{aligned}\right\} \tag{12.3}$$

where

$$J_2 = \int x^2\,dF, \quad J_3 = \int x^3\,dF, \quad dF = h\,dx.$$

Substitution of (11.15) and (12.3) in (11.7) yields:

$$\left.\begin{aligned} a_{11}W_1^{\mathrm{IV}} - 2\rho_{11}^0 W_1'' + s_{11}^0 W_1 = 0, \\ a_{22}W_2^{\mathrm{IV}} + [N_{22} - 2(b_{22} + \rho_{22}^0)]\,W_2'' + s_{22}^0 W_2 = 0, \end{aligned}\right\} \tag{12.4}$$

where for a plate of uniform thickness:

$$N_{22} = R\left(\frac{b^2}{12} - e_x^2\right). \tag{12.5}$$

The first equation (12.4) is independent of the load and therefore:

$$W_1 = 0.$$

Buckling of the plate is thus determined by the second equation (12.4), in which W_2 appears. Since the generalized displacement W_2 represents a rotation, torsional buckling will take place. It follows that no flexural buckling occurs in a prestressed plate.

Assuming that the lateral edges of the plate are simply supported, the solution of the second equation (12.4) has the form:

$$W_2 = C \sin\frac{\pi x}{l}. \tag{12.6}$$

Substitution of (12.6) in the second equation (12.4) gives the following value for the critical force:

$$R_{\mathrm{cr}} = \frac{1}{\frac{b^2}{12} - e_x^2}\left[a_{22}\frac{\pi^2}{l^2} + 2(b_{22} + \rho_{22}^0) + s_{22}^0\frac{l^2}{\pi^2}\right], \tag{12.7}$$

where a_{22}, b_{22}, ρ_{22}^0, s_{22}^0 are given by (11.15).

2

Consider a prestressed plate of uniform thickness, subjected to a compressive load applied at an eccentricity e_p (Figure 162).

The differential equations of buckling in this general case, which is a combination of the two previous ones, are:

$$\left.\begin{aligned} &a_{11}W_1^{IV} + (P - 2p_{11}^0)\, W_1'' + s_{11}^0 W_1 + P e_p W_2'' = 0,\\ &P e_p W_1'' + a_{22}W_2^{IV} + \left[(P + R)\frac{b^2}{12} - R e_x^2 - 2\,(b_{22} + \rho_{22}^0)\right] W_2'' +\\ &\qquad\qquad\qquad\qquad\qquad + s_{22}^0 W_2 = 0.\end{aligned}\right\} \qquad (12.8)$$

In the particular case when the external load is applied centrally ($e_p = 0$), the system of equations (12.8) can be separated into two independent equations, corresponding to flexural and to torsional buckling respectively.

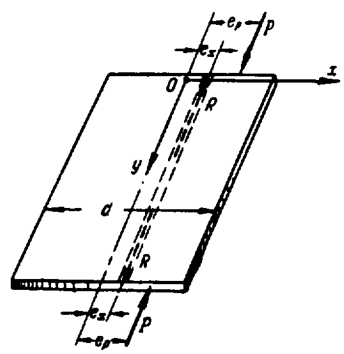


FIGURE 162.

3

In all the above examples it was assumed that the plate is simply supported at the lateral edges $y = 0$ and $y = l$. With other methods of support (built-in edges, free edges, etc.), the critical force is determined from the general integral of the corresponding homogeneous differential equation satisfying the boundary conditions. This yields a system of homogeneous algebraic equations in the integration constants, since the boundary conditions are also homogeneous in buckling problems. Equating to zero the determinant of this system (considering only the nontrivial solution) we obtain a transcendental equation in the parameter characterizing the compressive load. This equation has an infinite number of roots, the smallest of which determines the critical value of the compressive forces.

Chapter VII

METHOD OF INITIAL FUNCTIONS. APPLICATION OF THE METHOD TO THE THEORY OF THICK PLATES AND TO THE THEORY OF ELASTIC FOUNDATIONS

§ 1. GENERAL SOLUTION OF THE THREE-DIMENSIONAL PROBLEM OF THE THEORY OF ELASTICITY

1

The general problem of the equilibrium of a solid isotropic elastic body undergoing small deformations is described in cartesian coordinates by the differential equations:

$$\left.\begin{aligned} &\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+a=0,\\ &\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+\frac{\partial\tau_{yx}}{\partial x}+b=0,\\ &\frac{\partial\sigma_z}{\partial z}+\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+c=0,\end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned} \sigma_x&=\frac{2G}{1-2\nu}\left[(1-\nu)\frac{\partial u}{\partial x}+\nu\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right],\\ \sigma_y&=\frac{2G}{1-2\nu}\left[(1-\nu)\frac{\partial v}{\partial y}+\nu\left(\frac{\partial w}{\partial z}+\frac{\partial u}{\partial x}\right)\right],\\ \sigma_z&=\frac{2G}{1-2\nu}\left[(1-\nu)\frac{\partial w}{\partial z}+\nu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right],\\ \tau_{xy}&=\tau_{yx}=G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\\ \tau_{yz}&=\tau_{zy}=G\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right),\\ \tau_{zx}&=\tau_{xz}=G\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right),\end{aligned}\right\} \tag{1.2}$$

where $\sigma_x, \sigma_y, \ldots, \tau_{yz}, \tau_{xz}$ = components of the stress tensor; u, v, w = components of displacement vector of point considered; a, b, c = components of vector of the body force per unit volume at this point; $G=\frac{E}{2(1+\nu)}$ = modulus of elasticity in shear; ν = Poisson's ratio*.

* Equations (1.2) were already given in section 6 of Chapter I. The magnitudes introduced there were different, being the elastic constants E_0 and ν_0 of a three-dimensional body (the elastic foundation), defined by (6.3) of Chapter I.

As already mentioned earlier, two methods of solving the general problem are used in the theory of elasticity, namely the method of displacements and the method of stresses. The first method, in which the basic functions are the displacements $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$, was used in the preceding chapters when considering the strains in the elastic foundation in the two-dimensional and three-dimensional cases. In the second method the basic functions are the stresses $\sigma_x = \sigma_x(x, y, z), \ldots, \tau_{xz} = \tau_{xz}(x, y, z)$.

In addition, it is also possible to apply a mixed method, as will be done by us in the solution of the general three-dimensional problem of the theory of elasticity.

Let the basic unknown functions be the displacements $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$ and the stresses $\tau_{xz}, \tau_{yz}, \sigma_z$. The components u, v, w of the displacement vector will be considered positive if they coincide with the positive directions of the coordinate axes x, y, z.

Similarly, the components $\tau_{xz}, \tau_{yz}, \sigma_z$ of the stress vector acting on an elementary surface, whose outer normal is directed along the z axis, will be considered positive if they coincide with the positive directions of the x, y, z axes respectively. To simplify the notation, the displacements u, v, w will be replaced henceforth by the magnitudes:

$$U = Gu, \quad V = Gv, \quad W = Gw, \tag{1.3}$$

also called displacements.

The unknown stresses will be denoted:

$$\tau_{xz} = X, \quad \tau_{yz} = Y, \quad \sigma_z = Z. \tag{1.4}$$

Eliminating between (1.1) and (1.2) the stresses $\sigma_x, \sigma_y, \tau_{xy} = \tau_{yx}$ we obtain the system of six fundamental equations of the mixed method. Substituting (1.3) and (1.4), these equations can be presented in the form:

$$\left.\begin{aligned}
\frac{\partial U}{\partial z} &= -\frac{\partial W}{\partial x} + X,\\
\frac{\partial V}{\partial z} &= -\frac{\partial W}{\partial y} + Y,\\
\frac{\partial W}{\partial z} &= -\frac{\nu}{1-\nu}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) + \frac{1-2\nu}{2(1-\nu)} Z,\\
\frac{\partial Z}{\partial z} &= -\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - c,\\
\frac{\partial Y}{\partial z} &= -\frac{1+\nu}{1-\nu}\frac{\partial^2 U}{\partial x\,\partial y} - \left(\frac{\partial^2 V}{\partial x^2} + \frac{2}{1-\nu}\frac{\partial^2 V}{\partial y^2}\right) - \frac{\nu}{1-\nu}\frac{\partial Z}{\partial y} - b,\\
\frac{\partial X}{\partial z} &= -\frac{1+\nu}{1-\nu}\frac{\partial^2 V}{\partial x\,\partial y} - \left(\frac{\partial^2 U}{\partial y^2} + \frac{2}{1-\nu}\frac{\partial^2 U}{\partial x^2}\right) - \frac{\nu}{1-\nu}\frac{\partial Z}{\partial x} - a.
\end{aligned}\right\} \tag{1.5}$$

The remaining stresses are:

$$\left.\begin{aligned}
\sigma_x &= \frac{2}{1-2\nu}\left[(1-\nu)\frac{\partial U}{\partial x} + \nu\left(\frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right)\right],\\
\sigma_y &= \frac{2}{1-2\nu}\left[(1-\nu)\frac{\partial V}{\partial y} + \nu\left(\frac{\partial W}{\partial z} + \frac{\partial U}{\partial x}\right)\right],\\
\tau_{xy} &= \tau_{yx} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}.
\end{aligned}\right\} \tag{1.6}$$

From (1.5) and the boundary conditions, the six unknown geometrical and statical magnitudes characterizing the states of strain and stress can be obtained.

3

The mixed method of representing the general equations of equilibrium of an isotropic elastic body can also be applied to dynamical problems of the theory of elasticity. The unknown functions U, V, W, X, Y, Z depend in this case on the variables x, y, z, t; in (1.5), the expressions for the inertia forces

$$-\frac{m}{G}\frac{\partial^2 U}{\partial t^2}, \quad -\frac{m}{G}\frac{\partial^2 V}{\partial t^2}, \quad -\frac{m}{G}\frac{\partial^2 W}{\partial t^2}, \tag{1.7}$$

have to be added.

§ 2. SOLVING THE EQUATIONS OF THE THEORY OF ELASTICITY BY THE METHOD OF INITIAL FUNCTIONS

1

Consider two planes in the body: the plane $z=0$ and a plane $z=\text{const}$, parallel to it. The part of the body included between these planes represents a layer of thickness $z=\text{const}$. When z is fixed the unknowns in (1.5) depend only on x and y. Thus, the magnitudes U, V, W, X, Y, Z determine the displacement and stress vectors at any point (x, y) of the fixed plane $z=\text{const}$. The magnitudes $U_0, V_0, W_0, X_0, Y_0, Z_0$ corresponding to $z=0$, will henceforth be called geometrical and statical initial functions.

The positive directions of displacements and stresses for points of the lower plane $z=\text{const}$ and of the upper plane $z=0$ are shown in Figure 163 (the z axis is directed downward).

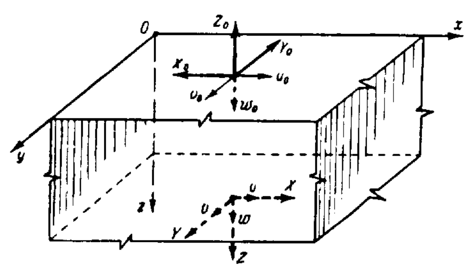


FIGURE 163.

For any plane $z=\text{const}$, the vector components are positive if they act along the positive directions of the coordinate axes. The same rule is also applied to the components U_0, V_0, W_0. The components X_0, Y_0, Z_0 are positive when their directions are opposed to those of the positive coordinates axes.

We assume a general solution of (1.5) in the form of Maclaurin series in z:

$$\left.\begin{aligned} U &= U_0 + z\left(\frac{\partial U}{\partial z}\right)_0 + \frac{z^2}{2!}\left(\frac{\partial^2 U}{\partial z^2}\right)_0 + \ldots \\ V &= V_0 + z\left(\frac{\partial V}{\partial z}\right)_0 + \frac{z^2}{2!}\left(\frac{\partial^2 V}{\partial z^2}\right)_0 + \ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ Z &= Z_0 + z\left(\frac{\partial Z}{\partial z}\right)_0 + \frac{z^2}{2!}\left(\frac{\partial^2 Z}{\partial z^2}\right)_0 + \ldots \end{aligned}\right\} \tag{2.1}$$

The following symbols will be used for the partial derivatives of any function $F = F(x, y, z)$:

$$\left.\begin{aligned} &\frac{\partial F}{\partial x} = \alpha F, & &\frac{\partial F}{\partial y} = \beta F, & &\frac{\partial F}{\partial z} = rF; \\ &\frac{\partial^2 F}{\partial x^2} = \alpha^2 F, & &\frac{\partial^2 F}{\partial y^2} = \beta^2 F, & &\frac{\partial^2 F}{\partial z^2} = r^2 F; \\ &\ldots\ldots & &\ldots\ldots & &\ldots\ldots \\ &\frac{\partial^n F}{\partial x^n} = \alpha^n F, & &\frac{\partial^n F}{\partial y^n} = \beta^n F, & &\frac{\partial^n F}{\partial z^n} = r^n F; \end{aligned}\right\} \tag{2.2}$$

in general:

$$\frac{\partial^{k+l+m} F}{\partial x^k \partial y^l \partial z^m} = \alpha^k \beta^l r^m F.$$

These symbols are those used in the so-called symbolic method, which makes possible the application of the methods of linear algebra to differentiation and transformation of equations.

We can then rewrite (1.5) as follows:

$$\left.\begin{aligned} rU &= -\alpha W + X, \\ rV &= -\beta W + Y, \\ rW &= -\frac{\nu}{1-\nu}(\alpha U + \beta V) + \frac{1-2\nu}{2(1-\nu)} Z, \\ rZ &= -\alpha X - \beta Y - c, \\ rY &= -\frac{1+\nu}{1-\nu}\alpha\beta U - \left(\alpha^2 V + \frac{2}{1-\nu}\beta^2 V\right) - \frac{\nu}{1-\nu}\beta Z - b, \\ rX &= -\frac{1+\nu}{1-\nu}\alpha\beta V - \left(\beta^2 U + \frac{2}{1-\nu}\alpha^2 U\right) - \frac{\nu}{1-\nu}\alpha Z - a. \end{aligned}\right\} \tag{2.3}$$

The body forces a, b, c will henceforth be assumed to vanish.

Multiplying (2.3) by r, and eliminating the terms containing $rU, rV, \ldots, rX$, we obtain the second derivatives with respect to z of the unknown functions:

$$\left.\begin{aligned} r^2 U &= -\left(\frac{2-\nu}{1-\nu}\alpha^2 + \beta^2\right) U - \frac{1}{1-\nu}\alpha\beta V - \frac{1}{2(1-\nu)}\alpha Z, \\ r^2 V &= -\frac{1}{1-\nu}\alpha\beta U - \left(\frac{2-\nu}{1-\nu}\beta^2 + \alpha^2\right) V - \frac{1}{2(1-\nu)}\beta Z, \\ r^2 W &= \frac{\nu}{1-\nu}(\alpha^2 + \beta^2) W - \frac{1}{2(1-\nu)}\alpha X - \frac{1}{2(1-\nu)}\beta Y, \end{aligned}\right\} \tag{2.4}$$

$$
\left.\begin{aligned}
r^2 Z &= \frac{2}{1-\nu}(\alpha^2+\beta^2)\,\alpha U + \frac{2}{1-\nu}(\alpha^2+\beta^2)\,\beta V + \frac{\nu}{1-\nu}(\alpha^2+\beta^2) Z,\\
r^2 Y &= \frac{2}{1-\nu}(\alpha^2+\beta^2)\,\beta W - \frac{1}{1-\nu}\alpha\beta X - \left(\frac{2-\nu}{1-\nu}\beta^2+\alpha^2\right) Y,\\
r^2 X &= \frac{2}{1-\nu}(\alpha^2+\beta^2)\,\alpha W - \left(\frac{2-\nu}{1-\nu}\alpha^2+\beta^2\right) X - \frac{1}{1-\nu}\alpha\beta Y.
\end{aligned}\right\} \qquad (2.4)
$$

The third derivatives with respect to z are obtained by multiplying (2.4) by r and eliminating the terms containing $rU, rV, \ldots, rX$ with the aid of (2.3). Higher derivatives are obtained in the same way.

3

Equations (2.3) and (2.4) are true for any values of the independent variables x, y, z. Putting $z=0$, we obtain the partial derivatives in the right-hand sides of (2.1). Grouping together the differential operations performed on the same functions ($U_0, V_0, \ldots, X_0$) we obtain the unknown functions $U, V, \ldots, X$, expressed through the initial functions $U_0, V_0, \ldots, X_0$ and their partial derivatives.

These formulas can be written in the form

$$
\left.\begin{aligned}
U &= L_{UU}U_0 + L_{UV}V_0 + \ldots + L_{UX}X_0,\\
V &= L_{VU}U_0 + L_{VV}V_0 + \ldots + L_{VX}X_0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
X &= L_{XU}U_0 + L_{XV}V_0 + \ldots + L_{XX}X_0,
\end{aligned}\right\} \qquad (2.5)
$$

where L_{UU}, $L_{UV}, \ldots, L_{XX}$ = linear differential operators with respect to the initial functions $U_0(x,y)$, $V_0(x,y), \ldots, X_0(x,y)$, depending on z and containing partial derivatives with respect to x and y for $z=0$. These operators can be represented as follows:

$$
\left.\begin{aligned}
L_{UU} = L_{XX} &= 1 - \frac{z^2(2-\nu)}{2(1-\nu)}\alpha^2 - \frac{z^2}{2}\beta^2 + \frac{z^4(3-\nu)}{24(1-\nu)}\gamma^2\alpha^2 +\\
&\quad + \frac{z^4}{24}\gamma^2\beta^2 - \frac{z^6(4-\nu)}{720(1-\nu)}\gamma^4\alpha^2 - \frac{z^6}{720}\gamma^4\beta^2 + \ldots\\
L_{UV} = L_{YX} &= -\frac{z^2}{2(1-\nu)}\alpha\beta + \frac{z^4}{12(1-\nu)}\gamma^2\alpha\beta -\\
&\quad - \frac{z^6}{240(1-\nu)}\gamma^4\alpha\beta + \ldots\\
L_{UW} = L_{ZX} &= -\alpha z + \frac{z^3(2-\nu)}{6(1-\nu)}\gamma^2\alpha - \frac{z^5(3-\nu)}{120(1-\nu)}\gamma^4\alpha +\\
&\quad + \frac{z^7(4-\nu)}{5040(1-\nu)}\gamma^6\alpha + \ldots\\
L_{UZ} = L_{WX} &= -\frac{z^2}{4(1-\nu)}\alpha + \frac{z^4}{24(1-\nu)}\gamma^2\alpha -\\
&\quad - \frac{z^6}{480(1-\nu)}\gamma^4\alpha + \frac{z^8}{20\,160(1-\nu)}\gamma^6\alpha + \ldots\\
L_{UY} = L_{VX} &= -\frac{z^3}{12(1-\nu)}\alpha\beta - \frac{z^5}{120(1-\nu)}\gamma^2\alpha\beta -\\
&\quad - \frac{z^7}{3360(1-\nu)}\gamma^4\alpha\beta + \ldots
\end{aligned}\right\} \qquad (2.6)
$$

$$
\left.
\begin{aligned}
L_{UX} &= z - \frac{z^3(3-2\nu)}{12(1-\nu)}\alpha^2 - \frac{z^3}{6}\beta^2 + \frac{z^5(2-\nu)}{120(1-\nu)}\gamma^2\alpha^2 + \\
&\quad + \frac{z^5}{120}\gamma^2\beta^2 - \frac{z^7(5-2\nu)}{10\,080(1-\nu)}\gamma^4\alpha^2 - \frac{z^7}{5040}\gamma^4\beta^2 + \ldots \\
L_{VU} = L_{XY} &= -\frac{z^2}{2(1-\nu)}\alpha\beta + \frac{z^4}{12(1-\nu)}\gamma^2\alpha\beta - \\
&\quad - \frac{z^6}{240(1-\nu)}\gamma^4\alpha\beta + \ldots \\
L_{VV} = L_{YY} &= 1 - \frac{z^2(2-\nu)}{2(1-\nu)}\beta^2 - \frac{z^2}{2}\alpha^2 + \frac{z^4(3-\nu)}{24(1-\nu)}\gamma^2\beta^2 + \\
&\quad + \frac{z^4}{24}\gamma^2\alpha^2 - \frac{z^6(4-\nu)}{720(1-\nu)}\gamma^4\beta^2 - \frac{z^6}{720}\gamma^4\alpha^2 + \ldots \\
L_{VW} = L_{ZY} &= -z\beta + \frac{z^3(2-\nu)}{6(1-\nu)}\gamma^2\beta - \frac{z^5(3-\nu)}{120(1-\nu)}\gamma^4\beta + \\
&\quad + \frac{z^7(4-\nu)}{5040(1-\nu)}\gamma^6\beta - \ldots \\
L_{VZ} = L_{WY} &= -\frac{z^2}{4(1-\nu)}\beta + \frac{z^4}{24(1-\nu)}\gamma^2\beta - \\
&\quad - \frac{z^6}{480(1-\nu)}\gamma^4\beta + \frac{z^8}{20\,160(1-\nu)}\gamma^6\beta - \ldots \\
L_{VY} &= z - \frac{z^3(3-2\nu)}{12(1-\nu)}\beta^2 - \frac{z^3}{6}\alpha^2 + \frac{z^5(2-\nu)}{120(1-\nu)}\gamma^2\beta^2 + \\
&\quad + \frac{z^5}{120}\gamma^2\alpha^2 - \frac{z^7(5-2\nu)}{10\,080(1-\nu)} - \frac{z^7}{5040}\gamma^4\alpha^2 + \ldots \\
L_{WU} = L_{XZ} &= -\frac{z\nu}{1-\nu}\alpha + \frac{z^3(1+\nu)}{6(1-\nu)}\gamma^2\alpha - \frac{z^5(2+\nu)}{120(1-\nu)}\gamma^4\alpha + \\
&\quad + \frac{z^7(3+\nu)}{5040(1-\nu)}\gamma^6\alpha - \ldots \\
L_{WV} = L_{YZ} &= -\frac{z\nu}{1-\nu}\beta + \frac{z^3(1+\nu)}{6(1-\nu)}\gamma^2\beta - \\
&\quad - \frac{z^5(2+\nu)}{120(1-\nu)}\gamma^4\beta + \frac{z^7(3+\nu)}{5040(1-\nu)}\gamma^6\beta - \ldots \\
L_{WW} = L_{ZZ} &= 1 + \frac{z^2\nu}{2(1-\nu)}\gamma^2 - \frac{z^4(1+\nu)}{24(1-\nu)}\gamma^4 + \\
&\quad + \frac{z^6(2+\nu)}{720(1-\nu)}\gamma^6 - \ldots \\
L_{WZ} &= \frac{z(1-2\nu)}{2(1-\nu)} + \frac{z^3\nu}{6(1-\nu)}\gamma^2 - \frac{z^5(1+2\nu)}{240(1-\nu)}\gamma^4 - \\
&\quad - \frac{z^7(1+\nu)}{5040(1-\nu)}\gamma^6 - \ldots \\
L_{ZU} = L_{XW} &= \frac{z^2}{1-\nu}\gamma^2\alpha - \frac{z^4}{6(1-\nu)}\gamma^4\alpha + \frac{z^6}{120(1-\nu)}\gamma^6\alpha - \ldots \\
L_{ZV} = L_{YW} &= \frac{z^2}{1-\nu}\gamma^2\beta - \frac{z^4}{6(1-\nu)}\gamma^4\beta + \frac{z^6}{120(1-\nu)}\gamma^6\beta - \ldots \\
L_{ZW} &= -\frac{z^3}{3(1-\nu)}\gamma^4 + \frac{z^5}{30(1-\nu)}\gamma^6 - \frac{z^7}{840(1-\nu)}\gamma^8 + \ldots \\
L_{YU} = L_{XV} &= -\frac{z(1+\nu)}{1-\nu}\alpha\beta + \frac{z^3(3+\nu)}{6(1-\nu)}\gamma^2\alpha\beta - \\
&\quad - \frac{z^5(5+\nu)}{120(1-\nu)}\gamma^4\alpha\beta + \frac{z^7(7+\nu)}{5040(1-\nu)}\gamma^6\alpha\beta - \ldots \\
L_{YV} &= -\frac{2z}{1-\nu}\beta^2 - z\alpha^2 + \frac{2z^3}{3(1-\nu)}\gamma^2\beta^2 + \frac{z^3}{6}\gamma^2\alpha^2 - \\
&\quad - \frac{z^5}{20(1-\nu)}\gamma^4\beta^2 - \frac{z^5}{120}\gamma^4\alpha^2 + \frac{z^7}{630(1-\nu)}\gamma^6\beta^2 + \frac{z^7}{5040}\gamma^6\alpha^2 - \ldots \\
L_{XU} &= -\frac{2z}{1-\nu}\alpha^2 - z\beta^2 + \frac{2z^3}{3(1-\nu)}\gamma^2\alpha^2 + \frac{z^3}{6}\gamma^2\beta^2 - \\
&\quad - \frac{z^5}{20(1-\nu)}\gamma^4\alpha^2 - \frac{z^5}{120}\gamma^4\beta^2 + \frac{z^7}{630(1-\nu)}\gamma^6\alpha^2 + \frac{z^7}{5040}\gamma^6\beta^2 - \ldots
\end{aligned}
\right\} \qquad (2.6)
$$

The symbols $\gamma^2, \gamma^4, \ldots, \gamma^{2n}$ denote two-dimensional harmonic, biharmonic, and n-harmonic differential operators in the x, y plane.

These operators are related to the single-term operators:

$$\alpha^2 = \frac{\partial^2}{\partial x^2}, \quad \beta^2 = \frac{\partial^2}{\partial y^2}$$

as follows:

$$\gamma^2 = \alpha^2 + \beta^2, \quad \gamma^4 = (\alpha^2 + \beta^2)^2, \ldots, \gamma^{2n} = (\alpha^2 + \beta^2)^n.$$

4

Considering the differential operators

$$\alpha, \ \alpha^2, \ \beta, \ \beta^2, \ \alpha\beta, \ \gamma^2, \ \gamma^4, \ldots, \gamma^{2n}$$

in the right-hand sides of (2.6) as algebraic magnitudes, i.e. performing on them the operations of addition, subtraction, multiplication, and division, we can represent the operators $L_{UU}, L_{UV}, L_{UW}, \ldots, L_{XY}, L_{XX}$ in (2.5) in closed form as trigonometric functions of the argument $\gamma z = z\sqrt{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}}$.

Using the series developments of the trigonometric functions:

$$\sin\varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \ldots, \qquad \cos\varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \ldots,$$

$$\frac{1}{2}(\sin\varphi - \varphi\cos\varphi) = \frac{\varphi^3}{3!} - \frac{2\varphi^5}{5!} - \frac{3\varphi^7}{7!} - \ldots,$$

$$\frac{1}{2}(\sin\varphi + \varphi\cos\varphi) = \varphi - \frac{2\varphi^3}{3!} + \frac{3\varphi^5}{5!} - \ldots,$$

$$\frac{1}{2}(\varphi\sin\varphi + 2\cos\varphi) = 1 - \frac{\varphi^4}{4!} + \frac{2\varphi^6}{6!} - \frac{3\varphi^8}{8!} + \ldots,$$

$$\frac{1}{2}(3\sin\varphi - \varphi\cos\varphi) = \varphi - \frac{\varphi^5}{5!} + \frac{2\varphi^7}{7!} - \frac{3\varphi^9}{9!} + \ldots$$

where we write $\varphi = \gamma z$, we can represent the series (2.6) in the form:

$$\left.\begin{aligned}
L_{XX} &= L_{UU} = \cos\gamma z - \frac{1}{2(1-\nu)}\frac{\alpha^2 z}{\gamma}\sin\gamma z,\\
L_{YX} &= L_{UV} = -\frac{1}{2(1-\nu)}\frac{\alpha\beta z}{\gamma}\sin\gamma z,\\
L_{ZX} &= L_{UW} = -\frac{1}{2(1-\nu)}\frac{\alpha}{\gamma}[(1-2\nu)\sin\gamma z + \gamma z\cos\gamma z],\\
L_{WX} &= L_{UZ} = -\frac{1}{4(1-\nu)}\frac{\alpha z}{\gamma}\sin\gamma z,\\
L_{VX} &= L_{UY} = -\frac{1}{4(1-\nu)}\frac{\alpha\beta}{\gamma^3}(\sin\gamma z - \gamma z\cos\gamma z),\\
L_{UX} &= \frac{1}{\gamma}\sin\gamma z - \frac{1}{4(1-\nu)}\frac{\alpha^2}{\gamma^3}(\sin\gamma z - \gamma z\cos\gamma z),\\
L_{XY} &= L_{VU} = -\frac{1}{2(1-\nu)}\frac{\alpha\beta z}{\gamma}\sin\gamma z,\\
L_{YY} &= L_{VV} = \cos\gamma z - \frac{1}{2(1-\nu)}\frac{\beta^2 z}{\gamma}\sin\gamma z,\\
L_{ZY} &= L_{\;W} = -\frac{1}{2(1-\nu)}\frac{\beta}{\gamma}[(1-2\nu)\sin\gamma z + \gamma z\cos\gamma z],
\end{aligned}\right| \qquad (2.7)$$

$$
\left.\begin{aligned}
L_{WY} &= L_{VZ} = -\frac{1}{4(1-\nu)}\frac{\beta z}{\gamma}\sin\gamma z,\\
L_{XZ} &= L_{WU} = \frac{1}{2(1-\nu)}\frac{\alpha}{\gamma}[(1-2\nu)\sin\gamma z-\gamma z\cos\gamma z],\\
L_{YZ} &= L_{WV} = \frac{1}{2(1-\nu)}\frac{\beta}{\gamma}[(1-2\nu)\sin\gamma z-\gamma z\cos\gamma z],\\
L_{ZZ} &= L_{WW} = -\frac{1}{1-\nu}\gamma(\sin\gamma z-\gamma z\cos\gamma z),\\
L_{WZ} &= \frac{1}{4(1-\nu)}\frac{1}{\gamma}[(3-4\nu)\sin\gamma z-\gamma z\cos\gamma z],\\
L_{XW} &= L_{ZU} = \frac{1}{1-\nu}\alpha\gamma z\sin\gamma z,\\
L_{YW} &= L_{ZV} = \frac{1}{1-\nu}\beta\gamma z\sin\gamma z,\\
L_{ZW} &= -\frac{1}{1-\nu}\gamma(\sin\gamma z-\gamma z\cos\gamma z),\\
L_{XV} &= L_{YU} = -\frac{1}{1-\nu}\frac{\alpha\beta}{\gamma}(\nu\sin\gamma z+\gamma z\cos\gamma z),\\
L_{YV} &= -\frac{\alpha^2}{\gamma}\sin\gamma z-\frac{1}{1-\nu}\frac{\beta^2}{\gamma}(\sin\gamma z+\gamma z\cos\gamma z),\\
L_{XU} &= -\frac{\beta^2}{\gamma}\sin\gamma z-\frac{1}{1-\nu}\frac{\alpha^2}{\gamma}(\sin\gamma z+\gamma z\cos\gamma z).
\end{aligned}\right\} \qquad (2.7)
$$

We have thus two forms of representing the differential operators $L_{UU}, L_{UV}, \ldots, L_{XX}$: a purely differential form given by the infinite series (2.6), and an integral-differential form, given by the transcendental equations (2.7), which contain operators of the form:

$$
\gamma = (\alpha^2+\beta^2)^{1/2} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^{1/2} \qquad \frac{1}{\gamma} = (\alpha^2+\beta^2)^{-1/2},
$$
$$
\frac{1}{\gamma^3} = (\alpha^2+\beta^2)^{-3/2},
$$

where, as before:

$$
\alpha^2+\beta^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.
$$

5

Using (2.5) and (2.6) to express $U(x,y,z)$, $V(x,y,z), \ldots, X(x,y,z)$ through $U_0(x,y), V_0(x,y), \ldots, X_0(x,y)$ and their partial derivatives, as well as through z, we obtain by means of (1.6) the remaining stresses σ_x, σ_y, and $\tau_{xy}=\tau_{yx}$, acting on surfaces parallel to the z axis:

$$
\left.\begin{aligned}
\sigma_x &= A_U U_0 + A_V V_0 + \ldots + A_X X_0,\\
\sigma_y &= B_U U_0 + B_V V_0 + \ldots + B_X X_0,\\
\tau_{xy} = \tau_{yx} &= C_U U_0 + C_V V_0 + \ldots + C_X X_0.
\end{aligned}\right\} \qquad (2.8)
$$

where

$$
\left.
\begin{aligned}
A_U &= \frac{2}{1-\nu}\alpha - \frac{z^2}{1-\nu}(2\alpha^2+\beta^2)\alpha + \frac{z^4}{12(1-\nu)}(3\alpha^2+\beta^2)\gamma^2\alpha - \\
&\qquad - \frac{z^6}{360(1-\nu)}(4\alpha^2+\beta^2)\gamma^4\alpha + \ldots, \\
A_V &= \frac{2\nu}{1-\nu}\beta - \frac{z^2}{1-\nu}[(1+\nu)\alpha^2+\nu\beta^2]\beta + \frac{z^4}{12(1-\nu)} \times \\
&\times [(2+\nu)\alpha^2+\nu\beta^2]\gamma^2\beta - \frac{z^6}{360(1-\nu)}[(3+\nu)\alpha^2+\nu\beta^2]\gamma^4\beta + \ldots, \\
A_W &= -\frac{2z}{1-\nu}(\alpha^2+\nu\beta^2) + \frac{z^3}{3(1-\nu)}(2\alpha^2+\nu\beta^2)\gamma^2 - \\
&- \frac{z^5}{60(1-\nu)}(3\alpha^2+\nu\beta^2)\gamma^4 + \frac{z^7}{2520(1-\nu)}(4\alpha^2+\nu\beta^2)\gamma^6 - \ldots, \\
A_Z &= \frac{\nu}{1-\nu} - \frac{z^2}{2(1-\nu)}[(1+\nu)\alpha^2+\nu\beta^2] + \frac{z^4}{24(1-\nu)} \times \\
&\times [(2+\nu)\alpha^2+\nu\beta^2]\gamma^2 - \frac{z^6}{720(1-\nu)}[(3+\nu)\alpha^2+\nu\beta^2]\gamma^4 + \ldots, \\
A_Y &= \frac{z\nu}{1-\nu}\beta - \frac{z^3}{6(1-\nu)}[(1+\nu)\alpha^2+\nu\beta^2]\beta + \frac{z^5}{120(1-\nu)} \times \\
&\times [(2+\nu)\alpha^2+\nu\beta^2]\gamma^2\beta - \frac{z^7}{5040(1-\nu)}[(3+\nu)\alpha^2+\nu\beta^2]\gamma^4\beta + \ldots, \\
A_X &= \frac{z(2-\nu)}{1-\nu}\alpha - \frac{z^3}{6(1-\nu)}[(3-\nu)\alpha^2+(2-\nu)\beta^2]\alpha + \\
&\qquad + \frac{z^5}{120(1-\nu)}[(4-\nu)\alpha^2+(2-\nu)\beta^2]\gamma^2\alpha - \\
&\qquad - \frac{z^7}{5040(1-\nu)}[(5-\nu)\alpha^2+(2-\nu)\beta^2]\gamma^4\alpha - \ldots, \\
B_U &= \frac{2\nu}{1-\nu}\alpha - \frac{z^2}{1-\nu}[\nu\alpha^2+(1+\nu)\beta^2]\alpha + \frac{z^4}{12(1-\nu)} \times \\
&\times [\nu\alpha^2+(2+\nu)\beta^2]\gamma^2\alpha - \frac{z^6}{360(1-\nu)}[\nu\alpha^2+(3+\nu)\beta^2]\gamma^4\alpha + \ldots, \\
B_V &= \frac{2}{1-\nu}\beta - \frac{z^2}{1-\nu}(\alpha^2+2\beta^2)\beta + \frac{z^4}{12(1-\nu)}(\alpha^2+3\beta^2)\gamma^2\beta - \\
&\qquad - \frac{z^6}{360(1-\nu)}(\alpha^2+4\beta^2)\gamma^4\beta + \ldots, \\
B_W &= -\frac{2z}{1-\nu}(\nu\alpha^2+\beta^2) + \frac{z^3}{3(1-\nu)}(\nu\alpha^2+2\beta^2)\gamma^2 - \\
&- \frac{z^5}{60(1-\nu)}(\nu\alpha^2+3\beta^2)\gamma^2 + \frac{z^7}{2520(1-\nu)}(\nu\alpha^2+4\beta^2)\gamma^6 - \ldots, \\
B_Z &= \frac{\nu}{1-\nu} - \frac{z^2}{2(1-\nu)} - \frac{z^2}{2(1-\nu)}[\nu\alpha^2+(1+\nu)\beta^2] + \\
&\qquad + \frac{z^4}{24(1-\nu)}[\nu\alpha^2+(2+\nu)\beta^2]\gamma^2 - \frac{z^6}{720(1-\nu)} \times \\
&\qquad\qquad \times [\nu\alpha^2+(3+\nu)\beta^2]\gamma^4 + \ldots, \\
B_Y &= \frac{z(2-\nu)}{1-\nu}\beta - \frac{z^3}{6(1-\nu)}[(2-\nu)\alpha^2+(3-\nu)\beta^2]\beta + \\
&\qquad + \frac{z^5}{120(1-\nu)}[(2-\nu)\alpha^2+(4-\nu)\beta^2]\gamma^2\beta - \\
&\qquad - \frac{z^7}{5040(1-\nu)}[(2-\nu)\alpha^2+(5-\nu)\beta^2]\gamma^4\beta + \ldots, \\
B_X &= \frac{z\nu}{1-\nu}\alpha - \frac{z^3}{6(1-\nu)}[\nu\alpha^2+(1+\nu)\beta^2]\alpha + \frac{z^5}{120(1-\nu)} \times \\
&\times [\nu\alpha^2+(2+\nu)\beta^2]\gamma^2\alpha - \frac{z^7}{5040(1-\nu)}[\nu\alpha^2+(3+\nu)\beta^2]\gamma^4\alpha + \ldots, \\
C_U &= \beta - \frac{z^2}{2}\left(\frac{3-\nu}{1-\nu}\alpha^2+\beta^2\right)\beta + \frac{z^4}{24}\left(\frac{5-\nu}{1-\nu}\alpha^2+\beta^2\right)\gamma^2\beta - \\
&\qquad - \frac{z^6}{720}\left(\frac{7-\nu}{1-\nu}\alpha^2+\beta^2\right)\gamma^4\beta + \ldots,
\end{aligned}
\right\} \tag{2.9}
$$

$$
\left.
\begin{aligned}
C_V &= \alpha - \frac{z^2}{2}\left(\alpha^2 + \frac{3-\nu}{1-\nu}\beta^2\right)\alpha + \frac{z^4}{24}\left(\alpha^2 + \frac{5-\nu}{1-\nu}\beta^2\right)\gamma^2\alpha - \\
&\qquad - \frac{z^6}{720}\left(\alpha^2 + \frac{7-\nu}{1-\nu}\beta^2\right)\gamma^4\alpha + \ldots, \\
C_W &= -2z\alpha\beta + \frac{z^3(1-\nu)}{3(1-\nu)}\gamma^2\alpha\beta - \frac{z^5(3-\nu)}{60(1-\nu)}\gamma^4\alpha\beta + \\
&\qquad + \frac{z^7(4-\nu)}{2520(1-\nu)}\gamma^6\alpha\beta - \ldots, \\
C_Z &= -\frac{z^2}{2(1-\nu)}\alpha\beta + \frac{z^4}{12(1-\nu)}\gamma^2\alpha\beta - \frac{z^6}{240(1-\nu)}\gamma^4\alpha\beta + \ldots, \\
C_Y &= z\alpha - \frac{z^3}{6}\left(\alpha^2 + \frac{2-\nu}{1-\nu}\beta^2\right)\alpha + \frac{z^5}{120}\left(\alpha^2 + \frac{3-\nu}{1-\nu}\beta^2\right)\gamma^2\alpha - \\
&\qquad - \frac{z^7}{5040}\left(\alpha^2 + \frac{4-\nu}{1-\nu}\beta^2\right)\gamma^4\alpha + \ldots, \\
C_X &= z\beta - \frac{z^3}{6}\left(\frac{2-\nu}{1-\nu}\alpha^2 + \beta^2\right)\beta + \frac{z^5}{120}\left(\frac{3-\nu}{1-\nu}\alpha^2 + \beta^2\right)\gamma^2\beta - \\
&\qquad - \frac{z^7}{5040}\left(\frac{4-\nu}{1-\nu}\alpha^2 + \beta^2\right)\gamma^4\beta + \ldots
\end{aligned}
\right\} \tag{2.9}
$$

Using (2.7) we can represent these operators in the following closed form:

$$
\left.
\begin{aligned}
A_U &= \frac{2}{1-\nu}\alpha\cos\gamma z - \frac{z\alpha^3}{\gamma(1-\nu)}\sin\gamma z, \\
A_W &= -\frac{(\alpha^3 + 2\nu\beta^2)}{(1-\nu)\gamma}\sin\gamma z - \frac{z\alpha^3}{1-\nu}\cos\gamma z, \\
A_V &= \frac{2\nu}{1-\nu}\beta\cos\gamma z - \frac{z\alpha^2\beta}{\gamma(1-\nu)}\sin\gamma z, \\
A_Z &= \frac{\nu}{1-\nu}\cos\gamma z - \frac{z}{2(1-\nu)\gamma}\alpha^2\sin\gamma z, \\
A_Y &= \frac{\beta}{\gamma(1-\nu)}\left(\nu - \frac{\alpha^2}{2\gamma^2}\right)\sin\gamma z + \frac{z\alpha^2\beta}{2(1-\nu)\gamma^2}\cos\gamma z, \\
A_X &= \frac{z\alpha^3}{2(1-\nu)\gamma^2}\cos\gamma z + \frac{\alpha}{(1-\nu)\gamma}\left[(2-\nu) - \frac{\alpha^2}{2\gamma^2}\right]\sin\gamma z, \\
B_U &= \frac{2\nu}{1-\nu}\alpha\cos\gamma z - \frac{z\alpha\beta^2}{(1-\nu)\gamma}\sin\gamma z, \\
B_W &= -\frac{2\nu\alpha^2 + \beta^2}{(1-\nu)\gamma}\sin\gamma z - \frac{z\beta^3}{1-\nu}\cos\gamma z, \\
B_V &= \frac{2\beta}{1-\nu}\cos\gamma z - \frac{z\beta^3}{\gamma(1-\nu)}\sin\gamma z, \\
B_Z &= \frac{\nu}{1-\nu}\cos\gamma z - \frac{z}{2(1-\nu)\gamma}\beta^2\sin\gamma z, \\
B_Y &= \frac{\beta}{\gamma(1-\nu)}\left[(2-\nu) - \frac{\beta^2}{2\gamma^2}\right]\sin\gamma z + \frac{z\beta^3}{2(1-\nu)\gamma^2}\cos\gamma z, \\
B_X &= \frac{\alpha}{\gamma(1-\nu)}\left(\nu - \frac{\beta^2}{2\gamma^2}\right)\sin\gamma z + \frac{z\alpha\beta^2}{3(1-\nu)\gamma^2}\cos\gamma z, \\
C_U &= \beta\cos\gamma z - \frac{z\alpha^2\beta}{(1-\nu)\gamma}\sin\gamma z, \\
C_W &= -\frac{\alpha\beta}{(1-\nu)\gamma}[(1-2\nu)\sin\gamma z + \gamma z\cos\gamma z], \\
C_V &= \alpha\cos\gamma z - \frac{z\alpha\beta^2}{(1-\nu)\gamma}\sin\gamma z, \\
C_Y &= \frac{\alpha}{\gamma}\sin\gamma z - \frac{1}{2(1-\nu)}\frac{\alpha\beta^2}{\gamma^3}(\sin\gamma z - \gamma z\cos\gamma z), \\
C_Z &= -\frac{\alpha\beta z}{2(1-\nu)\gamma}\sin\gamma z \\
C_X &= \frac{\beta}{\gamma}\sin\gamma z - \frac{\alpha^2\beta}{3(1-\nu)\gamma^3}(\sin\gamma z - \gamma z\cos\gamma z).
\end{aligned}
\right\} \tag{2.10}
$$

These expressions could also have been obtained directly from (1.5), (2.5), and (2.7).

§ 3. BASIC PROPERTIES OF THE LINEAR TRANSFORMATION MATRICES IN THE METHOD OF INITIAL FUNCTIONS

1

Equations (2.5) represent a general solution of the three-dimensional problem of the theory of elasticity. When the operators $L_{UU}, L_{UV}, \ldots, L_{XX}$ are defined either by the infinite series (2.6) or by the transcendental equations (2.7), we obtain a one-to-one correspondence between the six initial functions $U_0(x, y), V_0(x, y), \ldots, X_0(x, y)$, corresponding to points of the plane $z=0$, and the six unknown functions $U(x, y), V(x, y), \ldots, X(x, y)$ corresponding to points of any fixed plane $z = \text{const}$.

Equations (2.5) thus represent the general law of transformation of the initial into the unknown functions. An identical transformation corresponds to a unit matrix whose principal diagonal consists of unit elements, all other elements being zero. This property follows also from (2.7).

TABLE 22

	U	V	W	Z	Y	X
U	L_{UU}	L_{UV}	L_{UW}	L_{UZ}	L_{UY}	L_{UX}
V	L_{VU}	L_{VV}	L_{VW}	L_{VZ}	L_{VY}	L_{VX}
W	L_{WU}	L_{WV}	L_{WW}	L_{WZ}	L_{WY}	L_{WX}
Z	L_{ZU}	L_{ZV}	L_{ZW}	L_{ZZ}	L_{ZY}	L_{ZX}
Y	L_{YU}	L_{YV}	L_{YW}	L_{YZ}	L_{YY}	L_{YX}
X	L_{XU}	L_{XV}	L_{XW}	L_{XZ}	L_{XY}	L_{XX}
σ_x	A_U	A_V	A_W	A_Z	A_Y	A_X
σ_y	B_U	B_V	B_W	B_Z	B_Y	B_X
$\tau_{xy} = \tau_{yx}$	C_U	C_V	C_W	C_Z	C_Y	C_X

The transformation of the initial into the unknown functions is called direct transformation. The set of 36 operators $L_{UU}, L_{UV}, \ldots, L_{XX}$ forms the matrix of this direct linear transformation, given in Tables 22 and 23. If $U, V, \ldots, X$ are considered as given and $U_0, V_0, \ldots, X_0$ as unknown in (2.5), we obtain the inverse transformation. In this case the problem reduces to integrating a system of six compatible partial differential equations of an infinitely high order in the limit.

This seemingly complex problem is solved very simply by taking into consideration the physical meaning of the method of initial functions. Taking any plane $z = \text{const}$ as initial, the functions $U, V, \ldots, X$ as given (transformable), and the functions $U_0, V_0, \ldots, X_0$ as unknown (transformed), we assign a negative value to the coordinate z in (2.5). Taking into account that the operators $L_{UU}, L_{UV}, \ldots, L_{XX}$ are even functions of z and thus retain their sign, while the remaining operators $L_{UW}, L_{UY}, \ldots, L_{XY}$, are odd functions of z and change sign, we obtain:

$$\left.\begin{aligned}
U_0 &= L_{UU}U + L_{UV}V - L_{UW}W + L_{UZ}Z - L_{UY}Y - L_{UX}X,\\
V_0 &= L_{VU}U + L_{VV}V - L_{VW}W + L_{VZ}Z - L_{VY}Y - L_{VX}X,\\
W_0 &= -L_{WU}U - L_{WV}V + L_{WW}W - L_{WZ}Z + L_{WY}Y + L_{WX}X,\\
Z_0 &= L_{ZU}U + L_{ZV}V - L_{ZW}W + L_{ZZ}Z - L_{ZY}Y - L_{ZX}X,\\
Y_0 &= -L_{YU}U - L_{YV}V + L_{YW}W - L_{YZ}Z + L_{YY}Y + L_{YX}X,\\
X_0 &= -L_{XU}U - L_{XV}V + L_{XW} - L_{XZ}Z - L_{XY}Y + L_{XX}X.
\end{aligned}\right\} \tag{3.1}$$

2

Substitution in (2.5) of the functions $U_0, V_0, \ldots, X_0$ defined by (3.1), transforms the former into identities. It follows that transformations (2.5) and (3.1) are orthogonal. This property, observed in problems concerning thin-walled bars and shells, and known from Krylov's method of initial parameters in the analysis of beams on elastic foundations, is expressed mathematically as follows: the sum of the products of the corresponding elements in a line of the direct transformation (2.5) and in a column of the inverse transformation (3.1) equals unity, provided line and column have the same ordinal number.

The determinant formed by the operators in transformations (2.5) or (3.1) is equal to unity. This property, just as the property of orthogonality of transformations (2.5) and (3.1), is strictly fulfilled in the limit, when the operators $L_{UU}, L_{UV}, \ldots, L_{XX}$ are defined by (2.7).

TABLE 23

	U	V	W	Z	Y	X
U	$\cos\gamma z - \frac{1}{2(1-\nu)} \times \frac{\alpha^2 z}{\gamma}\sin\gamma z$	$-\frac{1}{2(1-\nu)} \times \frac{\alpha\beta z}{\gamma}\sin\gamma z$	$-\frac{\alpha}{2(1-\nu)\gamma} \times [(1-2\nu) \times \sin\gamma z + \gamma z\cos\gamma z]$	$-\frac{\alpha z}{4(1-\nu)\gamma} \times \sin\gamma z$	$-\frac{\alpha\beta}{4(1-\nu)\gamma^3} \times (\sin\gamma z - \gamma z\cos\gamma z)$	$\frac{1}{\gamma}\sin\gamma z - \frac{1}{4(1-\nu)}\frac{\alpha^2}{\gamma^3} \times (\sin\gamma z - \gamma z\cos\gamma z)$
V	$-\frac{\alpha\beta z}{2(1-\nu)\gamma}\sin\gamma z$	$\cos\gamma z - \frac{\beta^2 z}{2(1-\nu)\gamma} \times \sin\gamma z$	$-\frac{\beta}{2(1-\nu)\gamma} \times [(1-2\nu) \times \sin\gamma z + \gamma z\cos\gamma z]$	$-\frac{\beta z}{4(1-\nu)\gamma} \times \sin\gamma z$	$\frac{1}{\gamma}\sin\gamma z - \frac{1}{4(1-\nu)} \times \frac{\beta^2}{\gamma^3}(\sin\gamma z - \gamma z\cos\gamma z)$	
W	$\frac{1}{2(1-\nu)}\frac{\alpha}{\gamma} \times [(1-2\nu)\sin\gamma z - \gamma z\cos\gamma z]$	$\frac{1}{2(1-\nu)}\frac{\beta}{\gamma} \times [(1-2\nu) \times \sin\gamma z - \gamma z\cos\gamma z]$	$\frac{1}{2(1-\nu)}[\gamma z\sin\gamma z - 2(1-\nu)\cos\gamma z]$	$\frac{1}{4(1-\nu)\gamma} \times [(3-4\nu)\sin\gamma z - \gamma z\cos\gamma z]$		
Z	$\frac{\alpha\gamma z}{1-\nu}\sin\gamma z$	$\frac{\beta\gamma z}{1-\nu}\sin\gamma z$	$\frac{\gamma}{1-\nu} \times (\gamma z\cos\gamma z - \sin\gamma z)$			
Y	$-\frac{\alpha\beta}{(1-\nu)\gamma} \times (\nu\sin\gamma z + \gamma z\cos\gamma z)$	$-\frac{\alpha^2}{\gamma}\sin\gamma z - \frac{\beta^2}{(1-\nu)\gamma} \times (\sin\gamma z + \gamma z\cos\gamma z)$				
X	$-\frac{\beta^2}{\gamma}\sin\gamma z - \frac{\alpha^2}{(1-\nu)\gamma} \times (\sin\gamma z + \gamma z\cos\gamma z)$					

In addition to the properties listed, the operators in transformation (2.5) obey the following law:

$$L_{VU} = L_{XY}, \quad L_{UV} = L_{YX}, \text{etc.} \tag{3.2}$$

Hence, the matrices (2.5) and (3.1) are symmetrical with respect to the secondary diagonal.

The equality of the operators L_{VU} and L_{UV}, and of the operators L_{XY} and L_{YX} symmetrical to them, is due to the isotropy of the elastic body with respect to the z axis. Thus:

$$\left.\begin{array}{l} L_{YU} = L_{XV},\ L_{ZU} = L_{XW},\ L_{WU} = L_{XZ},\ L_{VU} = L_{XY}, \\ L_{UU} = L_{XX},\ L_{UV} = L_{YX},\ L_{UW} = L_{ZX},\ L_{UZ} = L_{WX},\ L_{UY} = L_{VX}. \end{array}\right\} \tag{3.3}$$

§ 4. GENERAL METHOD OF REDUCING THE THREE-DIMENSIONAL PROBLEM OF THE THEORY OF ELASTICITY TO A TWO-DIMENSIONAL PROBLEM

The six initial two-dimensional functions $U_0(x, y)$, $V_0(x, y), \ldots, X_0(x, y)$ are obtained by integrating (1.5) by the method of expanding the unknown functions in powers of z. The initial functions are determined by the boundary conditions for $z = 0$ and $z = h = \text{const}$ or, in the general case, for $z = h(x, y)$. These functions are determined at each of these planes. The boundary conditions may be purely statical, purely geometrical, or mixed.

In the case of statical boundary conditions, three components of the stress vector are given at the boundary surface. The unknown functions are in this case the components $U_0(x, y)$, $V_0(x, y)$, $W_0(x, y)$ of the displacement vector of the plane $z = 0$. A system of three differential equations for these functions is obtained from the statical boundary conditions at $z = h(x, y)$.

In the case of purely geometrical conditions, the displacement components are given, the stress components $X_0(x, y)$, $Y_0(x, y)$, $Z_0(x, y)$ being unknown. A system of three linear differential equations for these three unknown functions is obtained from the geometrical boundary conditions at $z = h(x, y)$.

In the mixed problem, the boundary conditions at $z = 0$ are given partly in displacements and partly in stresses. Three conditions altogether are given for each point of the plane $z = 0$. Three differential equations for the remaining three unknown functions are obtained from the three conditions at $z = h(x, y)$.

Expanding, according to the general method of initial functions, the boundary conditions for $z = 0$ and $z = h(x, y)$, we can always reduce the three-dimensional problem of the theory of elasticity to a two-dimensional problem described by a system of three linear differential equations in three unknown initial functions of x and y *.

These equations will have variable coefficients in the case of an elastic layer of variable thickness $h = h(x, y)$. If the thickness is constant, the coefficients will also be constant.

The order of the differential equations depends on the number of terms retained in (2.6).

* A similar method, though formulated differently, was proposed by A.N. Lur'e /56/.

The boundary conditions for $z = 0$ and $z = h(x, y)$ are satisfied exactly during the reduction of the three-dimensional to a two-dimensional problem. The boundary conditions at the lateral cylindrical surface are satisfied when integrating the differential equations of the two-dimensional problem. These conditions are satisfied up to the terms of (2.6) which have been discarded.

If in (2.6) we use only terms linear in z for the displacements, up to z^2 for the shearing stresses X and Y, and the first terms in z^3 for the normal stress Z, we obtain a solution which satisfies the boundary conditions on the lateral surface only in Saint-Venant's sense.

We arrive in this case at the general moment theory of thick plates, independent of Kirchhoff and Love's hypothesis. If terms of higher order are retained in (2.6), a more accurate theory of thick plates is obtained. In this case there appears on the lateral surface, in addition to the axial forces and moments considered in problems of plane stress and bending of a plate, also an equilibrium system of stresses, which can be reduced to generalized forces of the same nature as bimoments.

It is thus possible to develop by the method of initial functions a general bimoment theory of thick plates and shells, independent of Kirchoff and Love's hypothesis, by means of which the boundary-value problem can be solved with the required accuracy.

§ 5. THICK PLATE SUBJECTED TO A LOAD SYMMETRICAL WITH RESPECT TO ITS MIDDLE PLANE

1

Let a plate of uniform thickness $2h$ be subjected to surface loads (normal and shearing forces in the general case) acting at the planes $z = \pm h$, symmetrically with respect to the middle plane of the plate (Figure 164).

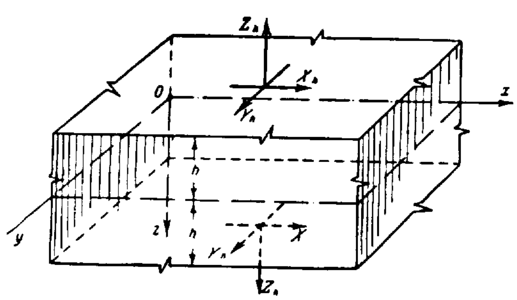


FIGURE 164.

We use the middle plane of the plate as reference plane. The z axis is directed downward, the x axis to the right, and the y axis in such way that the coordinate system xyz is right-handed. Due to symmetry, there will be no vertical displacements and shearing stresses in the middle plane of the plate; the three functions W_0, X_0, Y_0 will therefore vanish. The unknown functions will be the horizontal displacements $U_0(x, y)$ and $V_0(x, y)$, and the normal stress $Z_0(x, y)$.

Inserting $W_0 = X_0 = Y_0 = 0$ into (2.5) yields:

$$\left.\begin{aligned} U &= L_{UU} U_0 + L_{UV} V_0 + L_{UZ} Z_0, \\ V &= L_{VU} U_0 + L_{VV} V_0 + L_{VZ} Z_0, \\ W &= L_{WU} U_0 + L_{WV} V_0 + L_{WZ} Z_0, \\ Z &= L_{ZU} U_0 + L_{ZV} V_0 + L_{ZZ} Z_0, \\ Y &= L_{YU} U_0 + L_{YV} V_0 + L_{YZ} Z_0, \\ X &= L_{XU} U_0 + L_{XV} V_0 + L_{XZ} Z_0. \end{aligned}\right\} \tag{5.1}$$

The unknown functions $U_0(x, y)$, $V_0(x, y)$, $Z_0(x, y)$ are found by solving the system of three differential equations, obtained from (5.1) by equating the stress components X, Y, Z for $z = h$ to the given functions $Z_h(x, y)$, $Y_h(x, y)$, $X_h(x, y)$:

$$\left.\begin{aligned} L_{ZU}(h) U_0 + L_{ZV}(h) V_0 + L_{ZZ}(h) Z_0 = Z_h, \\ L_{YU}(h) U_0 + L_{YV}(h) V_0 + L_{YZ}(h) Z_0 = Y_h, \\ L_{XU}(h) U_0 + L_{XV}(h) V_0 + L_{XZ}(h) Z_0 = X_h, \end{aligned}\right\} \tag{5.2}$$

where $L_{ZU}(h), L_{ZV}(h), \ldots, L_{XZ}(h)$ = differential operators determined from (2.6) for $z = h$. When X_h, Y_h, Z_h are known, (5.2) forms a system of compatible partial differential equations in x and y.

2

The equilibrium of a plate subjected only to a normal load $Z_h(x, y)$, symmetrical with respect to the middle plane, will now be considered in more detail. The last two equations (5.2) are in this case homogeneous ($X_h = Y_h = 0$) and will be satisfied if we introduce the function $F = F(x, y)$ satisfying the equations:

$$\left.\begin{aligned} U_0 &= \quad (L_{XV} L_{YZ} - L_{YV} L_{XZ})_h F, \\ V_0 &= -(L_{XU} L_{YZ} - L_{YU} L_{XZ})_h F, \\ Z_0 &= \quad (L_{XU} L_{YV} - L_{YU} L_{XV})_h F, \end{aligned}\right\} \tag{5.3}$$

where the differential operators in parentheses are formed by the rules of symbolic differentiation for $z = h$. Substituting (5.3) in the first equation (5.2) we obtain:

$$\begin{aligned} [L_{ZU}(L_{XV} L_{YZ} - L_{YV} L_{XZ}) - L_{ZV}(L_{XU} L_{YZ} - L_{YU} L_{XZ}) + \\ + L_{ZZ}(L_{XU} L_{YV} - L_{YU} L_{XV})]_h F = Z_h, \end{aligned} \tag{5.4}$$

where the differential operator in brackets is determined approximately by (2.6) and exactly by (2.7), when $z = h$ is substituted.

The order of this equation depends on the number of terms taken in (2.6), which in turn depends on the relative thickness of the plate and the required accuracy of the solution. If for a plate of medium thickness only the first terms are taken in (2.6), we obtain the approximate theory of the equilibrium of a symmetrically loaded plate.

To obtain the exact theory, the exact values of the differential operators determined from (2.7) for $z=h$ should be substituted in (5.4). In this case, we obtain for F a transcendental equation in which the arguments of the trigonometric functions contain partial derivatives of F with respect to x and y. This equation can be written in the form:

$$\frac{\gamma^2}{1-\nu}\sin\gamma h\,[\sin\gamma h\cos\gamma h+\gamma h]\,F=Z_h. \tag{5.5}$$

Furthermore:

$$\left.\begin{aligned}U_0&=\frac{\alpha\sin\gamma h}{2(1-\nu)}[(1-2\nu)\sin\gamma h-\gamma h\cos\gamma h]\,F,\\V_0&=\frac{\beta\sin\gamma h}{2(1-\nu)}[(1-2\nu)\sin\gamma h-\gamma h\cos\gamma h]\,F,\\Z_0&=\frac{\gamma^2\sin\gamma h}{1-\nu}(\sin\gamma h+\gamma h\cos\gamma h)\,F.\end{aligned}\right\} \tag{5.6}$$

The order of (5.5) can be reduced by writing:

$$\Phi=\frac{\gamma\sin\gamma h}{1-\nu}F. \tag{5.7}$$

Equations (5.5) and (5.6) then become:

$$\gamma\left[\gamma h+\frac{\sin 2\gamma h}{2}\right]\Phi=Z_h, \tag{5.8}$$

$$\left.\begin{aligned}U_0&=\frac{\alpha}{2}\left[(1-2\nu)\frac{\sin\gamma h}{\gamma}-h\cos\gamma h\right]\Phi,\\V_0&=\frac{\beta}{2}\left[(1-2\nu)\frac{\sin\gamma h}{\gamma}-h\cos\gamma h\right]\Phi,\\Z_0&=\gamma\,[\sin\gamma h+\gamma h\cos\gamma h]\,\Phi.\end{aligned}\right\} \tag{5.9}$$

If the trigonometric functions in (5.8) and (5.9) are expanded in powers of their argument, the transcendental equation (5.8) becomes an ordinary differential equation. Writing again $\gamma^2=\alpha^2+\beta^2=\nabla^2$, we obtain:

$$\left[2h\nabla^2-\frac{2}{3}h^3\nabla^2\nabla^2+\frac{2}{15}h^5\nabla^2\nabla^2\nabla^2-\ldots\right]\Phi=Z_h. \tag{5.10}$$

Expressions (5.9) can then be written as follows:

$$\left.\begin{aligned}U_0&=\alpha h\left[-\nu+\frac{1+\nu}{6}h^2\nabla^2-\frac{2+\nu}{120}h^4\nabla^2\nabla^2+\ldots\right]\Phi,\\V_0&=\beta h\left[-\nu+\frac{1+\nu}{6}h^2\nabla^2-\frac{2+\nu}{120}h^4\nabla^2\nabla^2+\ldots\right]\Phi,\\Z_0&=h\ \left[2-\frac{2}{3}h^2\nabla^2+\frac{1}{20}h^4\nabla^2\nabla^2-\ldots\right]\nabla^2\Phi.\end{aligned}\right\} \tag{5.11}$$

Equations (5.8) or (5.10) describe exactly the states of strain and stress of a symmetrically loaded thick plate. After the function Φ has been determined from these equations and the boundary conditions on the lateral surface of the plate, the initial functions U_0, V_0, Z_0 can be obtained from (5.11), while the displacements U, V, W and the stresses Z, Y, X are found from (5.1). The remaining stresses σ_x, σ_y, τ_{xy} are then determined from (2.8).

An approximate solution is obtained by taking a finite number of terms in (5.10) and (5.11) or, which is the same, in (2.6). Thus, retaining only the first two terms in (5.10), we obtain:

$$\nabla^2\nabla^2\Phi - \frac{3}{h^2}\nabla^2\Phi = -\frac{3}{2h^3}Z_h. \tag{5.12}$$

The unknown initial functions are in this case:

$$U_0 = -\nu h\frac{\partial\Phi}{\partial x}, \quad V_0 = -\nu h\frac{\partial\Phi}{\partial y}, \quad Z_0 = 2h\nabla^2\Phi. \tag{5.13}$$

If the load acting on the plate is axisymmetrical, an ordinary differential equation in polar coordinates is obtained in both the exact and the approximate solution.

§ 6. THICK PLATE SUBJECTED TO A LOAD ANTISYMMETRICAL WITH RESPECT TO THE MIDDLE PLANE

1

If a plate of thickness $2h$ is subjected to a load consisting of normal and shearing stresses (Figure 165), applied antisymmetrically with respect to the middle plane $z=0$ at the boundary planes $z=\pm h$, the horizontal displacements and the normal stress at the middle plane will be equal to zero. Taking $z=0$ as reference plane, and putting in (2.5) $U_0=V_0=Z_0=0$, we obtain:

$$\left.\begin{aligned} U&=L_{UW}W_0+L_{UY}Y_0+L_{UX}X_0, & Z&=L_{ZW}W_0+L_{ZY}Y_0+L_{ZX}X_0,\\ V&=L_{VW}W_0+L_{VY}Y_0+L_{VX}X_0, & Y&=L_{YW}W_0+L_{YY}Y_0+L_{YX}X_0,\\ W&=L_{WW}W_0+L_{WY}Y_0+L_{WX}X_0, & X&=L_{XW}W_0+L_{XY}Y_0+L_{XX}X_0.\end{aligned}\right\} \tag{6.1}$$

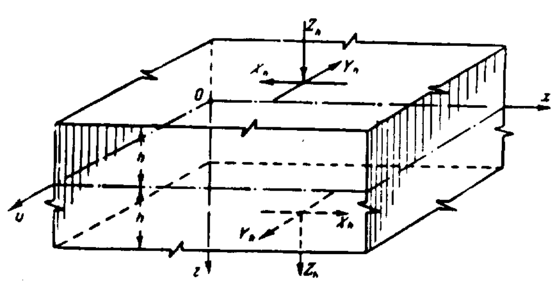


FIGURE 165.

The unknown initial functions are in this case the displacement $W_0 = W_0(x, y)$ and the stresses $X_0 = X_0(x, y)$ and $Y_0 = Y_0(x, y)$.
The following system of three differential equations is obtained for these three unknown functions from the statical boundary conditions:

$$\left.\begin{aligned} L_{XW} W_0 + L_{XX} X_0 + L_{XY} Y_0 &= X_h, \\ L_{YW} W_0 + L_{YX} X_0 + L_{YY} Y_0 &= Y_h, \\ L_{ZW} W_0 + L_{ZX} X_0 + L_{ZY} Y_0 &= Z_h, \end{aligned}\right\} \tag{6.2}$$

where $z = h$ has to be substituted in the operators $L_{XW}, L_{XX}, \ldots, L_{ZY}$.
If only a vertical load $Z_h = Z_h(x, y)$ acts on the plate, the first two equations (6.2) will be homogeneous. These equations can be satisfied by introducing a function $F = F(x, y)$ which satisfies the equations:

$$\left.\begin{aligned} W_0 &= (L_{XX} L_{YY} - L_{YX} L_{XY})_h F, \\ X_0 &= -(L_{XW} L_{YY} - L_{YW} L_{XY})_h F, \\ Y_0 &= (L_{XW} L_{YX} - L_{YW} L_{XX})_h F, \end{aligned}\right\} \tag{6.3}$$

(The subscript h indicates that the differential operators in parentheses are determined for $z = h$).
Substitution of (6.3) in the third equation (6.2) yields:

$$\begin{aligned} [L_{ZW} (L_{XX} L_{YY} - L_{YX} L_{XY}) - L_{ZX} (L_{XW} L_{YY} - L_{YW} L_{XY}) + \\ + L_{ZY} (L_{XW} L_{YX} - L_{YW} L_{XW})]_h F = Z_h. \end{aligned} \tag{6.4}$$

2

The order of (6.3) and (6.4) depends on the required degree of accuracy.
Expanding the differential operators in (6.3) according to (2.6) and substituting $z = h$ yields:

$$\left.\begin{aligned} W_0 &= \left[1 - \frac{h^2(2-\nu)}{2(1-\nu)} \nabla^2 + \frac{h^4(3-\nu)}{24(1-\nu)} \nabla^2\nabla^2 - \ldots\right] F, \\ X_0 &= \left[-\frac{h^2}{1-\nu} \nabla^2 + \frac{h^4}{6(1-\nu)} \nabla^2\nabla^2 - \ldots\right] \alpha F, \\ Y_0 &= \left[-\frac{h^2}{1-\nu} \nabla^2 + \frac{h^4}{6(1-\nu)} \nabla^2\nabla^2 - \ldots\right] \beta F, \end{aligned}\right\} \tag{6.5}$$

where, in accordance with the symbolic notation used:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \alpha F = \frac{\partial F}{\partial x}, \quad \beta F = \frac{\partial F}{\partial y}. \tag{6.6}$$

The following differential equation is obtained for the function F:

$$\begin{aligned} \left[\frac{2h^3}{3(1-\nu)} - \frac{2h^5}{15(1-\nu)} \nabla^2 + \frac{4h^7}{315(1-\nu)} \nabla^2\nabla^2 - \right. \\ \left. - \frac{2h^9}{2835(1-\nu)} \nabla^2\nabla^2\nabla^2 + \ldots\right] \nabla^2\nabla^2 F = Z_h. \end{aligned} \tag{6.7}$$

The fundamental equation of the problem considered is (6.7), which determines the function $F=F(x, y)$. The order of this equation depends on the required degree of accuracy.

To obtain an approximate solution, we retain only the first terms in (6.5) and (6.7), obtaining:

$$\left.\begin{aligned} W_0=F, \quad X_0=-\frac{h^2}{1-\nu}\nabla^2\frac{\partial F}{\partial x}, \qquad Y_0=-\frac{h^2}{1-\nu}\nabla^2\frac{\partial F}{\partial y},\\ \nabla^2\nabla^2 F=\frac{3(1-\nu)}{2h^3}Z_h. \end{aligned}\right\} \tag{6.8}$$

Writing, in accordance with (1.3):

$$W_0=\frac{E}{2(1+\nu)}w,$$

where $w=w(x, y)$ = actual vertical displacement of the points of the middle plane, and eliminating $F(x, y)$ from (6.8), we obtain:

$$\left.\begin{aligned} X_0=-\frac{Eh^2}{2(1-\nu^2)}\nabla^2\frac{\partial w}{\partial x}, \qquad Y_0=-\frac{Eh^2}{2(1-\nu^2)}\nabla^2\frac{\partial w}{\partial y},\\ \nabla^2\nabla^2 w=\frac{3(1-\nu^2)}{Eh^3}Z_h. \end{aligned}\right\} \tag{6.9}$$

Equations (6.9) and (6.8) correspond to the moment theory of the bending of plates which is a particular case of the general bimoment theory which is independent of Kirchhoff and Love's hypothesis. The moment theory holds true for sufficiently thin plates and distributed antisymmetrical loads. If the thickness of the plate is not small in relation to its other dimensions, and if the plate is subjected to local (concentrated) loads, the more general bimoment theory corresponding to (6.7) has to be applied. When the plate is of medium thickness, the first two or three terms (depending on the problem and the required accuracy) in (6.7) will be sufficient. The fundamental function $F=F(x, y)$ is invariant with respect to coordinate transmations.

The exact transcendental form of (6.4) and (6.7) is:

$$\frac{\gamma}{1-\nu}[\gamma h-\sin\gamma h\cos\gamma h]\,F=Z_h, \tag{6.10}$$

while (6.5) takes the form:

$$\left.\begin{aligned} W_0=\left[\cos\gamma h-\frac{\gamma h}{2(1-\nu)}\sin\gamma h\right]F,\\ Y_0=-\frac{\beta\gamma h}{1-\nu}\sin\gamma h F, \qquad X_0=-\frac{\alpha\gamma h}{1-\nu}\sin\gamma h F. \end{aligned}\right\} \tag{6.11}$$

§ 7. DEFORMATION OF AN ELASTIC FOUNDATION, DUE TO A LOAD APPLIED TO ITS SURFACE

1

Consider an elastic layer of finite thickness H, lying on an incompressible base and subjected to normal and shearing surface forces Z_H, X_H, Y_H (Figure 166). It will be assumed that at the plane of contact of this layer with the subsoil, the shearing stresses X and Y, and the vertical displacements W vanish. This means that the elastic layer can slide freely along the contact surface, as shown schematically in Figure 166.

Taking the plane of contact as reference plane, we again obtain expressions (5.1) for the displacements and stresses of the elastic layer. The problem considered is thus identical with the problem of a thick plate subjected to a symmetrical load.

In the general case, when surface forces Z_H, X_H, Y_H are present, the system of differential equations determining the solution is written in form (5.2). In the absence of shearing loads ($X_H = Y_H = 0$), we obtain again:

$$[L_{ZU}(L_{XV}L_{YZ} - L_{YV}L_{XZ}) - L_{ZV}(L_{XU}L_{YZ} - L_{YU}L_{XZ}) + L_{ZZ}(L_{XU}L_{YV} - L_{YU}L_{XV})]_H F = -Z_H \tag{7.1}$$

or

$$\left[2H\nabla^2 - \frac{2}{3}H^3\nabla^2\nabla^2 + \frac{2}{15}H^5\nabla^2\nabla^2\nabla^2 - \ldots\right]\Phi = -Z_H. \tag{7.2}$$

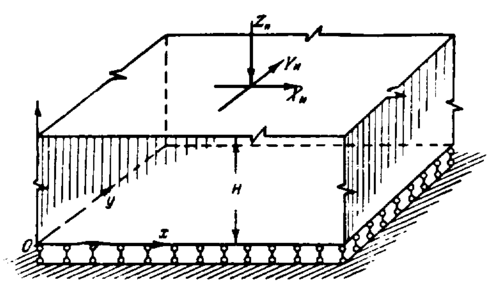


FIGURE 166.

Retaining only the first terms in (7.2), we obtain the differential equation of the approximate theory of an elastic foundation of finite thickness H:

$$\nabla^2\nabla^2\Phi - \frac{3}{H^2}\nabla^2\Phi + \frac{3}{2H^3}Z_H = 0, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{7.3}$$

The foundation model described by (7.3) corresponds in its behavior better to the elastic layer than the single-layer model considered before, since both vertical and horizontal displacements are taken into account.

Let the elastic layer, subjected to surface loads Z_H, X_H, Y_H, be fixed rigidly along the reference plane $z=0$ (Figure 167). In this case, the three initial functions U, V_0, W_0 will be equal to zero. The unknown functions will be the normal and shearing stresses Z_0, Y_0, X_0.

We obtain from (2.5):

$$\left.\begin{aligned} U &= L_{UZ} Z_0 + L_{UY} Y_0 + L_{UX} X_0, \\ V &= L_{VZ} Z_0 + L_{VY} Y_0 + L_{VX} X_0, \\ W &= L_{WZ} Z_0 + L_{WY} Y_0 + L_{WX} X_0, \\ Z &= L_{ZZ} Z_0 + L_{ZY} Y_0 + L_{ZX} X_0, \\ Y &= L_{YZ} Z_0 + L_{YY} Y_0 + L_{YX} X_0, \\ X &= L_{XZ} Z_0 + L_{XY} Y_0 + L_{XX} X_0 \end{aligned}\right\} \qquad (7.4)$$

The functions $Z_0(x, y)$, $Y_0(x, y)$, $X_0(x, y)$ are determined from the boundary conditions at $z=h$:

$$\left.\begin{aligned} L_{ZZ}(H) Z_0 + L_{ZY}(H) Y_0 + L_{ZX}(H) X_0 &= -Z_H, \\ L_{YZ}(H) Z_0 + L_{YY}(H) Y_0 + L_{YX}(H) X_0 &= Y_H, \\ L_{XZ}(H) Z_0 + L_{XY}(H) Y_0 + L_{XX}(H) X_0 &= X_H. \end{aligned}\right\} \qquad (7.5)$$

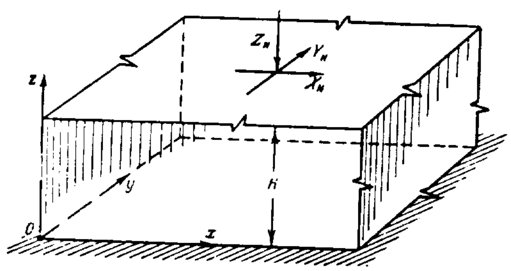


FIGURE 167.

This system of three partial differential equations in x and y represents the solving system of the problem considered.

In the particular case when only normal surface forces Z_H act on the elastic layer ($X_H = Y_H = 0$), we introduce the function $F = F(x, y)$ satisfying the equations:

$$\left.\begin{aligned} Z_0 &= (L_{XY} L_{YX} - L_{YY} L_{XX})_H F, \\ Y_0 &= -(L_{XZ} L_{YX} - L_{YZ} L_{XX})_H F, \\ X_0 &= (L_{XZ} L_{YY} - L_{YZ} L_{XY})_H F. \end{aligned}\right\} \qquad (7.6)$$

Substitution of (7.6) in (7.5) transforms the last two equations (7.5) into identities, and the first one into:

$$\begin{aligned} [L_{ZZ}(L_{XY} L_{YX} - L_{YY} L_{XX}) - L_{ZY}(L_{XZ} L_{YX} - L_{YZ} L_{XX}) + \\ + L_{ZX}(L_{XZ} L_{YY} - L_{YZ} L_{XY})]_H F = -Z_H. \end{aligned} \qquad (7.7)$$

This equation describes the states of stress and strain of a foundation rigidly fixed along the plane $z=0$. The order of this equation depends on the number of terms taken in (2.6). Substitution in (7.7) of the exact values of the differential operators, given by formulas (2.7) for $z=H$, yields:

$$\left[\frac{H^2}{4(1-\nu)^2}\gamma^2-\cos^2\gamma H-\frac{(1-2\nu)^2}{4(1-\nu)^2}\sin^2\gamma H\right]F=-Z_H, \tag{7.8}$$

while (7.6) becomes:

$$\left.\begin{aligned} Z_0&=\cos\gamma H\left[\frac{H\gamma}{2(1-\nu)}\sin\gamma H-\cos\gamma H\right]F,\\ Y_0&=\frac{\beta\cos\gamma H}{2\gamma(1-\nu)}\left[\gamma H\cos\gamma H-(1-2\nu)\sin\gamma H\right]F,\\ X_0&=\frac{\alpha\cos\gamma H}{2\gamma(1-\nu)}\left[(1-2\nu)\sin\gamma H-\gamma H\cos\gamma H\right]F.\end{aligned}\right\} \tag{7.9}$$

§ 8. CONTACT BETWEEN A PLATE AND AN ELASTIC FOUNDATION

Consider a plate subjected to a distributed load $p(x,\ y)$ and resting on an elastic foundation representing a compressible layer of finite thickness H (Figure 168).

The plane along which the elastic foundation rests on the underlying subsoil is taken as reference plane. We assume that the displacements at $z=0$ vanish: $U_0=V_0=W_0=0$. The states of stress and strain of the elastic foundation are then given by (7.4).

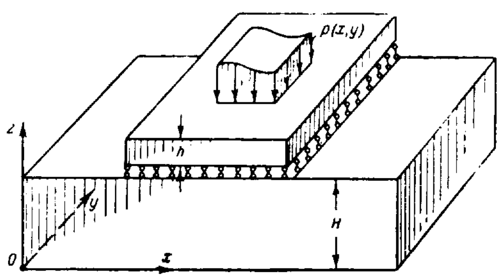


FIGURE 168.

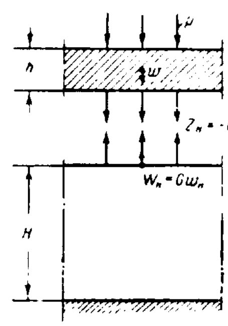


FIGURE 169.

The functions $Z_0,\ Y_0,\ X_0$ are determined from the boundary conditions at $z=H$. Assuming that there is no friction or adhesion between plate and elastic foundation, we obtain:

$$X_H=Y_H=0. \tag{8.1}$$

The differential equation of bending of the plate on the elastic foundation is:

$$D\nabla^2\nabla^2 w(x,\ y)=p(x,\ y)-q(x,\ y). \tag{8.2}$$

where $p(x, y)$ = given distributed load, $q(x, y)$ = reactions of elastic foundation.

According to our assumptions, the plate deflections $w(x, y)$ equal the vertical displacements of the elastic-foundation surface $w_H(x, y) = \frac{1}{G} W_H(x, y)$. It follows that $W_H = Gw$ (the positive directions of the deflections and displacements are shown in Figure 169). Furthermore, the reactions $q(x, y)$ represent a surface load $Z_H(x, y)$ with respect to the elastic foundation. In accordance with the convention adopted for the signs, the normal stresses at the surface of the elastic foundation are:

$$Z_H(x, y) = -\frac{D}{G}\nabla^2\nabla^2 W_H(x, y) - p(x, y). \tag{8.3}$$

Substitution of (7.4) in (8.1) and (8.3) yields:

$$\left.\begin{aligned}
&L_{ZZ}(H)Z_0 + L_{ZY}(H)Y_0 + L_{ZX}(H)X_0 = \\
&\quad = -\frac{D}{G}\nabla^2\nabla^2[L_{WZ}(H)Z_0 + L_{WY}(H)Y_0 + L_{WX}(H)X_0] - p, \\
&L_{YZ}(H)Z_0 + L_{YY}(H)Y_0 + L_{YX}(H)X_0 = 0, \\
&L_{XZ}(H)Z_0 + L_{XY}(H)Y_0 + L_{XX}(H)X_0 = 0.
\end{aligned}\right\} \tag{8.4}$$

or

$$\left.\begin{aligned}
&\left[L_{ZZ}(H) + \frac{D}{G}\gamma^4 L_{WZ}(H)\right]Z_0 + \left[L_{ZY}(H) + \frac{D}{G}\gamma^4 L_{WY}(H)\right]Y_0 + \\
&\qquad + \left[L_{ZX}(H) + \frac{D}{G}\gamma^4 L_{WX}(H)\right]X_0 = -p, \\
&L_{YZ}(H)Z_0 + L_{YY}(H)Y_0 + L_{YX}(H)X_0 = 0, \\
&L_{XZ}(H)Z_0 + L_{XY}(H)Y_0 + L_{XX}(H)X_0 = 0.
\end{aligned}\right\} \tag{8.5}$$

The differential operators $L_{ZZ}(H)$, $L_{WZ}(H)$, ..., $L_{XX}(H)$ are defined by (2.6) or (2.7) for $z = H$.

Introducing the function $F(x, y)$ satisfying (7.6), system (8.5) is reduced to the single equation:

$$\begin{aligned}
&\Big[\Big(L_{ZZ} + \frac{D}{G}\gamma^4 L_{WZ}\Big)(L_{XY}L_{YX} - L_{YY}L_{XX}) - \\
&\qquad - \Big(L_{ZY} + \frac{D}{G}\gamma^4 L_{WY}\Big)(L_{XZ}L_{YX} - L_{YZ}L_{XX}) + \\
&\qquad + \Big(L_{ZX} + \frac{D}{G}\gamma^4 L_{WX}\Big)(L_{XZ}L_{YY} - L_{YZ}L_{XY})\Big]_H F = -p.
\end{aligned} \tag{8.6}$$

This is the exact equation of bending of a plate resting on an elastic foundation considered as an isotropic layer of finite thickness H. Approximate solutions are obtained by taking a finite number of terms in (2.6), the order of (8.6) depending on this number, i.e., on the accuracy required of the solution.

§ 9. THEORY OF PLATES AND SHELLS OF VARIABLE THICKNESS, SUBJECTED TO ARBITRARY SURFACE LOADS

Consider the general equilibrium problem of a plate of variable thickness $h = h(x, y)$. This problem has considerable practical importance in the design of shallow shell-type roofings of variable thickness, having plane upper surfaces (Figure 170).

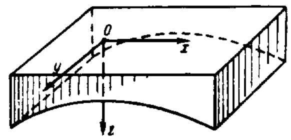


FIGURE 170.

Choosing such a surface as reference plane ($z=0$) and considering the surface load acting on it to be given, we obtain the stress components X_0, Y_0, Z_0 in (2.5) as known functions of x and y. The unknown initial functions in this region are the three displacements

$$U_0 = U_0(x, y), \quad V_0 = V_0(x, y), \quad W_0 = W(x, y).$$

Hence, the displacements

$$U = U(x, y, z), \quad V = V(x, y, z), \quad W = W(x, y, z)$$

and the stresses

$$X = X(x, y, z), \quad Y = Y(x, y, z), \quad Z = Z(x, y, z),$$
$$\sigma_x = \sigma_x(x, y, z), \quad \sigma_y = \sigma_y(x, y, z), \quad \tau_{xy} = \tau_{xy}(x, y, z)$$

at any point x, y, z are determined except for the three unknown initial functions $U_0(x, y)$, $V_0(x, y)$, $W_0(x, y)$. Substituting $z = h(x, y)$ in the general solution, we obtain the three components of the displacement vector and the six different components of the stress tensor for the points of the surface $h = h(x, y)$ forming the lower surface of the plate or shell. At $z = h(x, y)$, the stresses X, Y, Z, (σ_x, σ_y, τ_{xy}) must be in equilibrium with the given surface load applied to the lower surface $h = h(x, y)$.

Denoting by X_ν, Y_ν, Z_ν the components of this given surface load in the fixed cartesian reference frame x, y, z, the equilibrium conditions of an elementary tetrahedron, whose inclined surface forms part of the boundary surface $h = h(x, y)$, can be represented in the following form:

$$X_\nu = \sigma_x \cos(\nu, x) + \tau_{xy}\cos(\nu, y) + X\cos(\nu, z) \qquad (x, y, z), \tag{9.1}$$

where

$$\left.\begin{aligned} \cos(\nu, x) &= \frac{\partial h}{\partial x}\,\frac{1}{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}}, \\ \cos(\nu, y) &= \frac{\partial h}{\partial y}\,\frac{1}{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}}, \\ \cos(\nu, z) &= -\frac{1}{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}}. \end{aligned}\right\} \tag{9.2}$$

= cosines of angles between outer normal to element of surface $h = h(xy)$, and coordinate axes x, y, z respectively.

For $h = h(x, y)$ the statical boundary conditions are given by (9.1), which after insertion of (9.2) become:

$$
\left.
\begin{aligned}
X_\nu &= \frac{1}{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}} \left(\sigma_x \frac{\partial h}{\partial x} + \tau_{xy} \frac{\partial h}{\partial y} - X\right), \\
Y_\nu &= \frac{1}{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}} \left(\tau_{yx} \frac{\partial h}{\partial x} + \sigma_y \frac{\partial h}{\partial y} - Y\right), \\
Z_\nu &= \frac{1}{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}} \left(\tau_{zx} \frac{\partial h}{\partial x} + \tau_{zy} \frac{\partial h}{\partial y} - Z\right).
\end{aligned}
\right\} \tag{9.3}
$$

When the initial functions X_0, Y_0, Z_0 are known we obtain, by substituting in (9.3) the stress values given by (2.5) and (2.8), a system of three linear partial differential equations with variable coefficients, for the unknown functions $U_0(x, y)$, $V_0(x, y)$, $W_0(x, y)$. The order of these equations depends on the number of terms retained in (2.6).

We shall consider in detail the moment theory of plates and shells of variable thickness $h = h(x, y)$, assuming that the displacement W is constant over the shell thickness (i.e., does not depend on z), and that the displacements U and V vary linearly. Furthermore, the law of variation of the shearing stresses X and Y is given by a parabola of the second degree in Z, and of the normal stress Z, by a cubic parabola [the stresses σ_x, σ_y, and τ_{xy} vary linearly]. The following approximations then obtained from (2.5) and (2.8):

$$
\left.
\begin{aligned}
U &= U_0 - z\frac{\partial W_0}{\partial x} + zY_0,\quad V = V_0 - z\frac{\partial W_0}{\partial y} + zY_0,\quad W = W_0, \\
X &= -z\left(\frac{2}{1-\nu}\frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial y^2}\right) - \frac{1+\nu}{1-\nu} z \frac{\partial^2 V_0}{\partial x\,\partial y} - \\
&\quad - \frac{2-\nu}{2(1-\nu)} z^2 \frac{\partial^2 X_0}{\partial x^2} - \frac{z^2}{2}\frac{\partial^2 X_0}{\partial y^2} - \frac{1}{2(1-\nu)} z^2 \frac{\partial^2 Y_0}{\partial x\,\partial y} + \\
&\quad + \frac{1}{1-\nu} z^2 \nabla^2 \frac{\partial W_0}{\partial x} + X_0 - \frac{\nu}{1-\nu} z \frac{\partial Z_0}{\partial x}, \\
Y &= -\frac{1+\nu}{1-\nu} z \frac{\partial^2 U_0}{\partial x\,\partial y} - z\left(\frac{2}{1-\nu}\frac{\partial^2 V_0}{\partial y^2} + \frac{\partial^2 V_0}{\partial x^2}\right) - \\
&\quad - \frac{2-\nu}{2(1-\nu)} z^2 \frac{\partial^2 Y_0}{\partial y^2} - \frac{z^2}{2}\frac{\partial^2 Y_0}{\partial x^2} - \frac{1}{2(1-\nu)} z^2 \frac{\partial^2 X_0}{\partial x\,\partial y} + \\
&\quad + \frac{1}{1-\nu} z^2 \nabla^2 \frac{\partial W_0}{\partial y} + Y_0 - \frac{\nu}{1-\nu} z \frac{\partial Z_0}{\partial y}, \\
Z &= \frac{1}{1-\nu} z^2 \nabla^2 \left(\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right) - \frac{1}{3(1-\nu)} z^3 \nabla^2 \nabla^2 W_0 - \\
&\quad - z\left(\frac{\partial X_0}{\partial x} + \frac{\partial Y_0}{\partial y}\right) + Z_0 + \frac{z^2\nu}{2(1-\nu)} \nabla^2 Z_0 + \\
&\quad + \frac{2-\nu}{6(1-\nu)} z^3 \nabla^2 \left(\frac{\partial X_0}{\partial x} + \frac{\partial Y_0}{\partial y}\right), \\
\sigma_x &= \frac{2}{1-\nu}\left(\frac{\partial U_0}{\partial x} + \nu\frac{\partial V_0}{\partial y}\right) - \frac{2}{1-\nu} z \left(\frac{\partial^2 W_0}{\partial x^2} + \nu \frac{\partial^2 W_0}{\partial y^2}\right) + \\
&\quad + \frac{2-\nu}{1-\nu} z \frac{\partial X_0}{\partial x} + \frac{\nu}{1-\nu} z \frac{\partial Y_0}{\partial y} + \frac{\nu}{1-\nu_0} Z_0, \\
\sigma_y &= \frac{2}{1-\nu}\left(\nu\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right) - \frac{2}{1-\nu} z \left(\nu\frac{\partial^2 W_0}{\partial x^2} + \frac{\partial^2 W_0}{\partial y^2}\right) + \\
&\quad + \frac{\nu}{1-\nu} z \frac{\partial X_0}{\partial x} + \frac{2-\nu}{1-\nu} z \frac{\partial Y_0}{\partial y} + \frac{\nu}{1-\nu} Z_0, \\
\tau_{xy} &= \frac{\partial U_0}{\partial y} + \frac{\partial V_0}{\partial x} - 2z\frac{\partial^2 W_0}{\partial x\,\partial y} + z\left(\frac{\partial X_0}{\partial y} + \frac{\partial Y_0}{\partial x}\right).
\end{aligned}
\right\} \tag{9.4}
$$

It is easily seen that the stresses given by (9.4) satisfy (1.1). Substitution of these values in (9.3) yields a system of differential equations for determining the unknown functions $U_0(x, y)$, $V_0(x, y)$, $W_0(x, y)$.

Equations (9.3) and (9.4) describe the general moment theory of a plate or shell of variable thickness $h = h(x, y)$. This theory, based on more general assumptions than Kirchhoff and Love's hypothesis that linear elements remain normal to the middle surface, makes it possible to determine the stresses and strains of a plate or shell for an arbitrary law of variation of its thickness, i.e., for any shape of the lower surface $h = h(x, y)$ of the shell. Equations (9.3) must be supplemented by the corresponding boundary conditions, given for the unknown functions U_0, V_0, W_0 in accordance with the model adopted.

The exact solution of this boundary-value problem for plates of variable thickness $h = h(x, y)$ is very difficult and can hardly be carried out by the methods available at present. Bubnov and Galerkin's variational method is the best existing method for the approximate integration of equations with variable coefficients.

§ 10. GENERAL SOLUTION OF THE TWO-DIMENSIONAL PROBLEM OF THE THEORY OF ELASTICITY

It was shown above that the solution by the method of initial functions of the general three-dimensional problem of the theory of elasticity reduces to determining the six initial functions U_0, V_0, W_0, X_0, Y_0, Z_0. Since the two-dimensional problem is a particular case of the general three-dimensional problem, four initial functions will be sufficient to determine the states of stress and strain of the body, these being the displacements $u_0(x)$, $v_0(x)$ and the stresses $\tau^0_{xy}(x)$, $\sigma^0_y(x)$ at $y = 0$ (Figure 171). This can be proved by taking the displacements $u(x, y)$, $v(x, y)$ and the stresses $\tau_{xy}(x, y)$, $\sigma_y(x, y)$ as unknowns, and representing them as infinite series in powers of y.

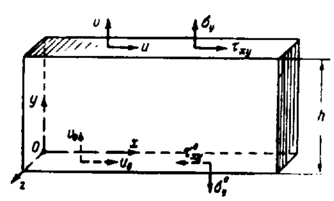


FIGURE 171.

In the two-dimensional case the equilibrium equations (1.1) of an elastic isotropic body become, when no volume forces act:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} = 0. \tag{10.1}$$

The relationships between stresses and displacements for the case of plane strain are:

$$\left.\begin{aligned}\sigma_x &= \frac{2G}{1-2\nu}\left[(1-\nu)\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}\right],\\ \sigma_y &= \frac{2G}{1-2\nu}\left[(1-\nu)\frac{\partial v}{\partial y}+\nu\frac{\partial u}{\partial x}\right],\\ \tau_{xy} &= \tau_{yx} = G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\end{aligned}\right\} \tag{10.2}$$

where

$$G=\frac{E}{2(1+\nu)}.$$

Introducing the symbols:

$$\left.\begin{aligned} U &= Gu, & V &= Gv,\\ X &= \tau_{xy}, & Y &= \sigma_y,\\ \frac{\partial}{\partial x} &= \alpha, & \frac{\partial}{\partial y} &= \beta,\end{aligned}\right\} \tag{10.3}$$

we can rewrite (10.1) and (10.2) in the form:

$$\left.\begin{aligned}\beta U &= -\alpha V + X,\\ \beta V &= -\frac{\nu}{1-\nu}\alpha U + \frac{1-2\nu}{2(1-\nu)}Y,\\ \beta Y &= -\alpha X,\\ \beta X &= -\frac{2}{1-\nu}\alpha^2 U - \frac{\nu}{1-\nu}\alpha V,\end{aligned}\right\} \tag{10.4}$$

whence

$$\sigma_x = \frac{2}{1-2\nu}\left[(1-\nu)\alpha U + \nu\beta V\right]. \tag{10.5}$$

Expanding, as in (2.1), the unknown magnitudes in Maclaurin series of powers of Y, we obtain the following solution of system (10.4):

$$\left.\begin{aligned} U &= L_{UU}U_0 + L_{UV}V_0 + L_{UY}Y_0 + L_{UX}X_0,\\ V &= L_{VU}U_0 + L_{VV}V_0 + L_{VY}Y_0 + L_{VX}X_0,\\ Y &= L_{YU}U_0 + L_{YV}V_0 + L_{YY}Y_0 + L_{YX}X_0,\\ X &= L_{XU}U_0 + L_{XV}V_0 + L_{XY}Y_0 + L_{XX}X_0,\end{aligned}\right\} \tag{10.6}$$

where L_{UU}, L_{UV}, ..., L_{XY}, L_{XX} are, as before, the linear differential operators on the initial functions $U_0(x)$, $V_0(x)$, $Y_0(x)$, $X_0(x)$; these operators, which are functions of y and contain derivatives with respect to x, can be represented either by infinite series (Table 24) or in transcendental form (Table 25). The bottom lines of Tables 24 and 25 give the operators obtained from (10.5) entering in the expression for σ_x:

$$\sigma_x = A_U U_0 + A_V V_0 + A_Y Y_0 + A_X X_0. \tag{10.7}$$

TABLE 24

	U_0	V_0	Y_0	X_0
U	$L_{UU} = 1 - \frac{y^2(2-\nu)}{2(1-\nu)}\alpha^2 + \frac{y^4(3-\nu)}{24(1-\nu)}\alpha^4 - \frac{y^6(4-\nu)}{720(1-\nu)}\alpha^6 + \ldots$	$L_{UV} = -y\alpha + \frac{y^3(2-\nu)}{6(1-\nu)}\alpha^3 - \frac{y^5(3-\nu)}{120(1-\nu)}\alpha^5 + \frac{y^7(4-\nu)}{5040(1-\nu)}\alpha^7 - \ldots$	$L_{UY} = -\frac{y^2}{4(1-\nu)}\alpha + \frac{y^4}{24(1-\nu)}\alpha^3 - \frac{y^6}{480(1-\nu)}\alpha^5 + \frac{y^8}{20160(1-\nu)}\alpha^7 - \ldots$	$L_{UX} = y - \frac{y^3(3-2\nu)}{12(1-\nu)}\alpha^2 + \frac{y^5(2-\nu)}{120(1-\nu)}\alpha^4 - \frac{y^7(5-2\nu)}{10080(1-\nu)}\alpha^6 + \ldots$
V	$L_{VU} = -\frac{y\nu}{1-\nu}\alpha + \frac{y^3(1+\nu)}{6(1-\nu)}\alpha^3 - \frac{y^5(2+\nu)}{120(1-\nu)}\alpha^5 + \frac{y^7(3+\nu)}{5040(1-\nu)}\alpha^7 - \ldots$	$L_{VV} = 1 + \frac{y^2\nu}{2(1-\nu)}\alpha^2 - \frac{y^4(1+\nu)}{24(1-\nu)}\alpha^4 + \frac{y^6(2+\nu)}{720(1-\nu)}\alpha^6 - \ldots$	$L_{VY} = \frac{y(1-2\nu)}{2(1-\nu)} + \frac{y^3\nu}{6(1-\nu)}\alpha^2 - \frac{y^5(1+2\nu)}{240(1-\nu)}\alpha^4 + \frac{y^7(1+\nu)}{5040(1-\nu)}\alpha^6 - \ldots$	$L_{VX} = L_{UY}$
Y	$L_{YU} = \frac{y^2}{1-\nu}\alpha^3 - \frac{y^4}{6(1-\nu)}\alpha^5 + \frac{y^6}{120(1-\nu)}\alpha^7 - \ldots$	$L_{YV} = -\frac{y^3}{3(1-\nu)}\alpha^4 + \frac{y^5}{30(1-\nu)}\alpha^6 - \frac{y^7}{840(1-\nu)}\alpha^8 + \ldots$	$L_{YY} = L_{VV}$	$L_{YX} = L_{UV}$
X	$L_{XU} = -\frac{2y}{1-\nu}\alpha^2 + \frac{2y^3}{3(1-\nu)}\alpha^4 - \frac{y^5}{20(1-\nu)}\alpha^6 + \frac{y^7}{630(1-\nu)}\alpha^8 - \ldots$	$L_{XV} = L_{YU}$	$L_{XY} = L_{VU}$	$L_{XX} = L_{UU}$
σ_x	$A_U = \frac{2}{1-\nu}\alpha - \frac{2y^2}{1-\nu}\alpha^3 + \frac{y^4}{4(1-\nu)}\alpha^5 - \frac{y^6}{90(1-\nu)}\alpha^7 + \ldots$	$A_V = -\frac{2y}{1-\nu}\alpha^2 + \frac{2y^3}{3(1-\nu)}\alpha^4 - \frac{y^5}{20(1-\nu)}\alpha^6 + \frac{y^7}{630(1-\nu)}\alpha^8 - \ldots$	$A_Y = \frac{\nu}{1-\nu} - \frac{y^2(1+\nu)}{2(1-\nu)}\alpha^2 + \frac{y^4(2+\nu)}{24(1-\nu)}\alpha^4 - \frac{y^6(3+\nu)}{720(1-\nu)}\alpha^6 + \ldots$	$A_X = \frac{y(2-\nu)}{1-\nu}\alpha - \frac{y^3(3-\nu)}{6(1-\nu)}\alpha^3 + \frac{y^5(4-\nu)}{120(1-\nu)}\alpha^5 - \frac{y^7(5-\nu)}{5040(1-\nu)}\alpha^7 + \ldots$

TABLE 25

	U_0	V_0	Y_0	X_0
U	$L_{UU} = \cos\alpha y -$ $- \frac{\alpha y}{2(1-\nu)}\sin\alpha y$	$L_{UV} = -\frac{1-2\nu}{2(1-\nu)}\sin\alpha y -$ $- \frac{\alpha y}{2(1-\nu)}\cos\alpha y$	$L_{UY} = -\frac{y}{4(1-\nu)}\sin\alpha y$	$L_{UX} = \frac{1}{\alpha}\sin\alpha y -$ $- \frac{1}{4(1-\nu)}\frac{1}{\alpha} \times$ $\times (\sin\alpha y - \alpha y\cos\alpha y)$
V	$L_{VU} = \frac{1-2\nu}{2(1-\nu)}\sin\alpha y -$ $- \frac{\alpha y}{2(1-\nu)}\cos\alpha y$	$L_{VV} = \frac{\alpha y}{2(1-\nu)} \times$ $\times \sin\alpha y + \cos\alpha y$	$L_{VY} = \frac{3-4\nu}{4(1-\nu)\alpha}\sin\alpha y -$ $- \frac{y}{4(1-\nu)}\cos\alpha y$	$L_{VX} = L_{UY}$
Y	$L_{YU} = \frac{\alpha^2 y}{1-\nu}\sin\alpha y$	$L_{YV} = -\frac{\alpha}{1-\nu} \times$ $\times (\sin\alpha y - \alpha y\cos\alpha y)$	$L_{YY} = L_{VV}$	$L_{YX} = L_{UV}$
X	$L_{XU} = -\frac{\alpha}{1-\nu} \times$ $\times (\sin\alpha y + \alpha y\cos\alpha y)$	$L_{XV} = L_{YU}$	$L_{XY} = L_{VU}$	$L_{XX} = L_{UU}$
σ_x	$A_U = \frac{2\alpha}{1-\nu}\cos\alpha y -$ $- \frac{y\alpha^2}{1-\nu}\sin\alpha y$	$A_V = -\frac{\alpha}{1-\nu} \times$ $\times (\sin\alpha y + y\alpha\cos\alpha y)$	$A_Y = \frac{\nu}{1-\nu}\cos\alpha y -$ $- \frac{y\alpha}{2(1-\nu)}\sin\alpha y$	$A_X = \frac{y\alpha}{2(1-\nu)}\cos\alpha y +$ $- \frac{3-2\nu}{2(1-\nu)}\sin\alpha y$

Equations (10.2) through (10.5) and Tables 24, 25 correspond to the case of plane strain. The corresponding expressions for plane stress are obtained from them by replacing the modulus of elasticity E and Poisson's ratio ν by $\frac{E(1+2\nu)}{(1+\nu)^2}$, and $\frac{\nu}{1+\nu}$, respectively. Performing this substitution in Table 25 yields the matrix of linear transformation of the functions $U_0(x)$ $V_0(x)$, $Y_0(x)$, $X_0(x)$ into the functions $U(x, y)$, $V(x, y)$, $Y(x, y)$, $X(x, y)$ for the case of plane stress, given in Table 26.

Equations (10.6) represent the law of transformation of the initial into the unknown functions and give the general solution of the two-dimensional problem of the theory of elasticity. These equations are symmetrical with respect to the secondary diagonal:

$$L_{UY} = L_{VX},\ L_{UV} = L_{YX},\ L_{VV} = L_{YY},\ L_{UU} = L_{XX},$$
$$L_{VU} = L_{XY},\ L_{YU} = L_{XV}.$$

The initial functions U_0, V_0, Y_0, X_0, which in (10.6) constitute four arbitrary functions obtained by integrating (10.4), are determined by the boundary conditions at $y = 0$ and $y = h = \text{const}$. Two functions can be prescribed for every plane $y = \text{const}$.

Since two initial functions are always known from the beginning, the solution of the two-dimensional problem reduces to the determination of two initial functions from the boundary conditions for $y = h$. These boundary conditions yield a system of two ordinary differential equations, which, in the general case, are of infinitely high order. When the problem is solved by approximation, the order of these equations depends on the number of terms retained in Table 24.

TABLE 26

	U_0	V_0	Y_0	X_0
U	$L_{UU} = \cos ay - \frac{1+\nu}{2} ay \sin ay$	$L_{UV} = -\frac{1}{2}[(1-\nu)\sin ay + (1+\nu) ay \cos ay]$	$L_{UY} = -\frac{1+\nu}{4} y \sin ay$	$L_{UX} = \frac{3-\nu}{4a} \sin ay + \frac{1+\nu}{4} y \cos ay$
V	$L_{VU} = \frac{1}{2}[(1-\nu)\sin ay - (1+\nu) ay \cos ay]$	$L_{VV} = \frac{1+\nu}{2} ay \sin ay + \cos ay$	$L_{VY} = L_{UX}$	$L_{VX} = L_{UY}$
Y	$L_{YU} = (1+\nu) a^2 y \sin ay$	$L_{YV} = (1+\nu) a \times (ay \cos ay - \sin ay)$	$L_{YY} = L_{VV}$	$L_{YX} = L_{UV}$
X	$L_{XU} = -(1+\nu) a \times (\sin ay + ay \cos ay)$	$L_{XV} = L_{YU}$	$L_{XY} = L_{VU}$	$L_{XX} = L_{UU}$
σ_x	$A_U = (1+\nu) a \times (2 \cos ay - ay \sin ay)$	$A_V = -(1+\nu) a \times (\sin ay + ay \cos ay)$	$A_Y = \nu \cos ay - \frac{1+\nu}{2} ay \sin ay$	$A_X = \frac{1}{2}[(1+\nu) ay \cos ay + (3+\nu) \sin ay]$

§ 11. BENDING OF A THICK PLATE IN THE CASE OF PLANE STRAIN. APPROXIMATE SOLUTION

Consider the bending of a thick plate in the case of plane strain (Figure 172). Let the external load consist only of normal forces $p(x)$ disposed antisymmetrically with respect to the middle surface $y = 0$.

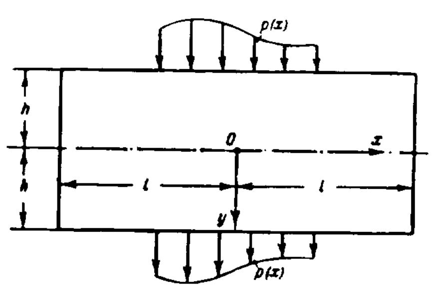


FIGURE 172.

Taking $y=0$ as reference plane, we obtain:

$$U_0 = Y_0 = 0, \tag{11.1}$$

so that (10.6) reduces to:

$$\left.\begin{aligned} U &= L_{UV}V_0 + L_{UX}X_0,\\ V &= L_{VV}V_0 + L_{VX}X_0,\\ Y &= L_{YV}V_0 + L_{YX}X_0,\\ X &= L_{XV}V_0 + L_{XX}X_0. \end{aligned}\right\} \tag{11.2}$$

From the boundary conditions for $y=h$, namely: $Y_h = p(x)$, $X_h = 0$, we obtain:

$$\left.\begin{aligned} L_{YV}(h)\,V_0 + L_{YX}(h)\,X_0 &= p,\\ L_{XV}(h)\,V_0 + L_{XX}(h)\,X_0 &= 0. \end{aligned}\right\} \tag{11.3}$$

We introduce the function F satisfying the equation:

$$L_{XX}(h)\,F = V_0, \qquad L_{XV}(h)\,F = -X_0 \tag{11.4}$$

Substitution of these expressions transforms the second equation (11.3) into an identity, while the first becomes:

$$(L_{YV}L_{XX} - L_{YX}L_{XV})_h F = p. \tag{11.5}$$

The solution is obtained by rewriting (11.5) either as ordinary differential equation, in which case the operators are given in Table 24, or as transcendental equation, Table 25 being used instead. The second method is more convenient, since transition from the transcendental integral-differential to the ordinary form is easy.

Substitution in (11.5) of the value given in Table 25 yields:

$$\frac{\alpha}{1-\nu}\,[\alpha h - \sin\alpha h\cos\alpha h]\,F = p, \tag{11.6}$$

while (11.4) becomes:

$$\left.\begin{aligned} V_0 &= \left[\cos\alpha h - \frac{1}{2(1-\nu)}\,\alpha h\sin\alpha h\right]F,\\ X_0 &= -\frac{1}{1-\nu}\,\alpha^2 h\sin\alpha h\cdot F. \end{aligned}\right\} \tag{11.7}$$

Expanding the trigonometric functions in (11.6) and (11.7) in power series, we obtain:

$$\frac{2h^3}{3(1-\nu)}\,\alpha^4\left[1 - \frac{h^2}{5}\alpha^2 + \frac{2h^4}{105}\alpha^4 - \frac{h^6}{945}\alpha^6 + \dots\right]F = p \tag{11.8}$$

$$\left.\begin{aligned} V_0 &= \left[1 - \frac{2-\nu}{2(1-\nu)}\,h^2\alpha^2 + \frac{3-\nu}{24(1-\nu)}\,h^4\alpha^4 - \frac{4-\nu}{720(1-\nu)}\,h^6\alpha^6 + \dots\right]F,\\ X_0 &= -\frac{h^2}{1-\nu}\,\alpha^3\left[1 - \frac{h^2}{6}\alpha^2 + \frac{h^4}{120}\alpha^4 - \frac{h^6}{5040}\alpha^6 + \dots\right]F. \end{aligned}\right\} \tag{11.9}$$

Equation (11.6) or (11.8) represents the exact solution of the problem considered. To obtain approximate solutions, we retain in (11.8) only a finite number of terms. Thus, if only the first term is used, (11.8) and (11.9) reduce to:

$$\frac{2h^3}{3(1-\nu)} F^{IV} = p, \tag{11.10}$$

$$\left.\begin{aligned} V_0 &= F, \\ X_0 &= -\frac{h^2}{1-\nu} F'''. \end{aligned}\right\} \tag{11.11}$$

Thus, in a first approximation, F equals V_0, i.e., the vertical displacements of the middle surface of the plate, while (11.10) becomes the ordinary equation of the bending of a beam in the case of plane strain. The first approximation thus yields the elementary solution corresponding to the hypothesis of plane sections. The matrix of the initial functions or, which is the same, of (11.2), is in this case given by Table 27.

TABLE 27

	V_0	X_0
U	$-y\alpha$	—
V	1	—
Y	$-\frac{y^3}{3(1-\nu)}\alpha^4$	$-y\alpha$
X	$\frac{y^2}{1-\nu}\alpha^3$	1
σ_x	$-\frac{2y}{1-\nu}\alpha^2$	—

It is seen that the horizontal displacements vary linearly with y; the vertical displacements are constant; the laws of variation of the normal stresses $\sigma_y = Y$ and the shearing stresses $\tau_{xy} = X$, are respectively parabolas of the third and second degree.

Substitution of (11.11) in Table 27 yields:

$$\left.\begin{aligned} &U = -yV_0', \qquad V = V_0, \\ &(1-\nu)Y = \frac{y}{3}(3h^2 - y^2)V_0^{IV}, \\ &(1-\nu)X = (y^2 - h^2)V_0''' \\ &(1-\nu)\sigma_x = -2yV_0''. \end{aligned}\right\} \tag{11.12}$$

The function V_0 is determined from (11.10) and the boundary conditions at the plate edges $x=\pm l$. Two conditions can be formulated at each edge, in agreement with (11.12) and (11.10). For a free edge, not under load, this solution makes it possible to eliminate the stresses only in the sense of Saint-Venant, i.e., by equating to zero the moment and the shearing force ($V_0''=0$, $V_0'''=0$).

In a second approximation we obtain from (11.8) and (11.9):

$$F^{VI}-\frac{5}{h^2}F^{IV}=-\frac{15}{2h^3}p(1-\nu), \tag{11.13}$$

$$\left.\begin{aligned} V_0&=F-\frac{2-\nu}{2(1-\nu)}h^2F'', \\ X_0&=-\frac{1}{1-\nu}h^2F''' + \frac{1}{6(1-\nu)}h^4F^{V}. \end{aligned}\right\} \tag{11.14}$$

We assume that the vertical displacements are constant:

$$V=V_0. \tag{11.15}$$

Substituting (11.14) and (11.1) in Table 24, and retaining (in accordance with the order of (11.13)) in the expression obtained for U, the terms containing F' and F''', in the expression for Y, the terms containing F^{IV} and F^{VI}, in the expression for X, the terms containing F''' and F^{V}, and in the expression for σ_x, the terms containing F'' and F^{IV}, we can represent the unknown displacements and stresses of the plate in the form:

$$\left.\begin{aligned} U&=-yF'+\frac{2-\nu}{6(1-\nu)}y\left[y^2-\frac{3\nu}{2-\nu}h^2\right]F''', \\ (1-\nu)Y&=\frac{y}{3}(3h^2-y^2)F^{IV}-\frac{y}{30}(5h^4-y^4)F^{VI}, \\ (1-\nu)X&=(y^2-h^2)F'''-\frac{y^4-h^4}{6}F^{V}, \\ (1-\nu)\sigma_x&=-2yF''+\frac{2}{3}y^3F^{IV}. \end{aligned}\right\} \tag{11.16}$$

The first terms in (11.16) are identical with (11.12). The additional terms take into account the deviation from the hypothesis of plane sections.

The general integral of (11.13) is:

$$F=C_1+C_2x+C_3x^2+C_4x^3+C_5\,\mathrm{sh}\frac{\sqrt{5}}{h}x+C_6\,\mathrm{ch}\frac{\sqrt{5}}{h}x+G, \tag{11.17}$$

where G = particular integral of (11.13), depending on external load, and $C_1,\ldots,C_6$ = constants.

To determine the six integration constants we require three boundary conditions at each lateral edge of the plate. For built-in edges ($V=0$, $U=0$), we obtain from (11.14), (11.15), and (11.16):

$$\left.\begin{aligned} &F-\frac{2-\nu}{2(1-\nu)}h^2F''=0, \\ &F'=0, \qquad F'''=0. \end{aligned}\right\} \tag{11.18}$$

If a membrane, rigid in its plane but flexible in bending, is placed at the edge ($V=0$, $\sigma_x=0$), the boundary conditions will be:

$$F=0, \quad F''=0, \quad F^{IV}=0. \tag{11.19}$$

Finally, for a free edge not under load ($\sigma_x=0$, $\tau_{xy}=0$), the boundary conditions will be:

$$\left.\begin{aligned} &F''=0, \quad F^{IV}=0,\\ &\int_0^h\left[(y^2-h^2)F'''-\frac{y^4-h^4}{6}F^V\right]dy=0. \end{aligned}\right\} \tag{11.20}$$

The plate can therefore be analyzed in a second approximation for any boundary conditions at the lateral edges $x=\pm l$. After determining the integration constants from these conditions, and then the function F from (11.17), we can find the displacements and stresses in the plate from (11.16). This procedure is applicable to thick plates, for which the deviation from the hypothesis of plane sections is considerable. If in (11.16) Poisson's ratio ν is replaced by $\frac{\nu}{1+\nu}$, we obtain the equation of bending of a high beam (beam-wall) for the case of plane stress.

Higher-order approximations for greater accuracy can be obtained by increasing the number of terms retained in the expansions. This, however, increases the order of the differential equations and makes their solution more laborious. The second approximation is quite satisfactory in practice.

In this section we have considered only the bending of a thick plate for arbitrary boundary conditions at its lateral edges $x=\pm l$, showing how this problem can be solved by approximations. The same procedure is possible in many other problems of plane stress or strain involving massive structures (see sections 5, 6, 7, and 8). In all these cases the fundamental solution is obtained from the boundary conditions at the longitudinal edges of the plate; an approximate solution is then obtained by retaining a number of terms depending on the accuracy required.

§ 12. USE OF TRIGONOMETRIC SERIES IN THE SOLUTION OF THE TWO-DIMENSIONAL PROBLEM*

1

We shall now consider problems of the theory of rectangular plates whose boundary conditions can be expressed with the aid of trigonometric series. Let the plate edges $x=0$ and $x=l$ (Figure 173) be rigidly connected to thin

* This and the following sections are based in part on V. V. Vlasov's Candidate's Thesis, Metod nachal'nykh funktsii v ploskoi zadache teorii uprugosti (The Method of Initial Functions in the Two-dimensional Problem of the Theory of Elasticity), 1958, and on his papers /13, 14/.

membranes perfectly rigid with respect to displacements in their plane, but freely displaceable out of their plane. These boundary conditions are formulated as follows:

at $x=0$ and $x=l$: $$V=Y=\sigma_x=0. \tag{12.1}$$

It follows from (12.1) and (10.1), (10.2) that

$$\left.\begin{aligned}
U(x,y)&=\sum_{n=1}^{\infty} f_{1n}(y)\cos\alpha_n x, & Y(x,y)&=\sum_{n=1}^{\infty} f_{3n}(y)\sin\alpha_n x,\\
V(x,y)&=\sum_{n=1}^{\infty} f_{2n}(y)\sin\alpha_n x, & X(x,y)&=\sum_{n=1}^{\infty} f_{4n}(y)\cos\alpha_n x,\\
\sigma_x(x,y)&=\sum_{n=1}^{\infty} f_{6n}(y)\sin\alpha_n x,
\end{aligned}\right\} \tag{12.2}$$

where $\alpha_n=\frac{n\pi}{l}$, l = plate length in x direction.

Equations (12.2) represent Filon's solution. If, on the other hand, U and X are expressed by sine series, and V, Y, and σ_x by cosine series, we obtain Ribière's solution satisfying the boundary conditions:

at $x=0$ and $x=l$: $$U=X=0. \tag{12.3}$$

We shall use (12.2), assuming all the initial functions U_0, V_0, X_0, and Y_0 to be known for $y=0$, being represented by trigonometric series with constant coefficients:

$$\left.\begin{aligned}
U_0&=\sum_{n=1}^{\infty} u_n\cos\alpha_n x, & Y_0&=\sum_{n=1}^{\infty} y_n\sin\alpha_n x,\\
V_0&=\sum_{n=1}^{\infty} v_n\sin\alpha_n x, & X_0&=\sum_{n=1}^{\infty} x_n\cos\alpha_n x.
\end{aligned}\right\} \tag{12.4}$$

The states of stress and strain of the plate can be expressed through the initial functions which satisfy (12.1), by substituting (12.4) in the general integrals of displacements and stresses, written in the form of Table 25. For example, the first term in the expression for U becomes:

$$\left(\cos\alpha_n y-\frac{1}{2(1-\nu)}\alpha_n y\sin\alpha_n y\right)U_0(x)=$$
$$=\sum_{n=1}^{\infty} u_n\sum_{m=0}^{\infty}\left[\frac{(\alpha_n y)^{2m}}{(2m)!}+\frac{1}{2(1-\nu)}\frac{(\alpha_n y)^{2m+2}}{(2m+1)!}\right]\cos\alpha_n x=$$
$$=\sum_{n=1}^{\infty} u_n\left(\operatorname{ch}\alpha_n y+\frac{1}{2(1-\nu)}\alpha_n y\operatorname{sh}\alpha_n y\right)\cos\alpha_n x.$$

The same procedure is applied to the other terms in Table 25. We obtain:

$$
\left.
\begin{aligned}
U &= \frac{1}{1-\nu}\sum_{n=1}^{\infty}\Big\{u_n\Big[(1-\nu)\operatorname{ch}\alpha_n y+\frac{1}{2}\alpha_n y\operatorname{sh}\alpha_n y\Big]-\\
&\quad-\frac{v_n}{2}[(1-2\nu)\operatorname{sh}\alpha_n y+\alpha_n y\operatorname{ch}\alpha_n y]-\frac{1}{4}y_n y\operatorname{sh}\alpha_n y+\\
&\quad+\frac{x_n}{4}\Big[\frac{3-4\nu}{\alpha_n}\operatorname{sh}\alpha_n y+y\operatorname{ch}\alpha_n y\Big]\Big\}\cos\alpha_n x.\\
V &= \frac{1}{1-\nu}\sum_{n=1}^{\infty}\Big\{\frac{u_n}{2}[\alpha_n y\operatorname{ch}\alpha_n y-(1-2\nu)\operatorname{sh}\alpha_n y]+\\
&\quad+v_n\Big[(1-\nu)\operatorname{ch}\alpha_n y-\frac{1}{2}\alpha_n y\operatorname{sh}\alpha_n y\Big]+\frac{y_n}{4}\Big[\frac{3-4\nu}{\alpha_n}\operatorname{sh}\alpha_n y-\\
&\quad-y\operatorname{ch}\alpha_n y\Big]+\frac{1}{4}x_n y\operatorname{sh}\alpha_n y\Big\}\sin\alpha_n x,\\
Y &= \frac{1}{1-\nu}\sum_{n=1}^{\infty}\Big\{u_n\alpha_n^2 y\operatorname{sh}\alpha_n y+v_n\alpha_n(\operatorname{sh}\alpha_n y-\alpha_n y\operatorname{ch}\alpha_n y)+\\
&\quad+y_n\Big[(1-\nu)\operatorname{ch}\alpha_n y-\frac{1}{2}\alpha_n y\operatorname{sh}\alpha_n y\Big]+\frac{x_n}{2}\Big[(1-2\nu)\operatorname{sh}\alpha_n y+\\
&\quad+\alpha_n y\operatorname{ch}\alpha_n y\Big]\Big\}\sin\alpha_n x,\\
X &= \frac{1}{1-\nu}\sum_{n=1}^{\infty}\Big\{u_n\alpha_n(\operatorname{sh}\alpha_n y+\alpha_n y\operatorname{ch}\alpha_n y)-v_n\alpha_n^2 y\operatorname{sh}\alpha_n y\\
&\quad+\frac{y_n}{2}[(1-2\nu)\operatorname{sh}\alpha_n y-\alpha_n y\operatorname{ch}\alpha_n y]+\\
&\quad+x_n\Big[(1-\nu)\operatorname{ch}\alpha_n y+\frac{1}{2}\alpha_n y\operatorname{sh}\alpha_n y\Big]\Big\}\cos\alpha_n x,\\
\sigma_x &= \frac{1}{1-\nu}\sum_{n=1}^{\infty}\Big\{-u_n\alpha_n(2\operatorname{ch}\alpha_n y+\alpha_n y\operatorname{sh}\alpha_n y)+\\
&\quad+v_n\alpha_n(\operatorname{sh}\alpha_n y+\alpha_n y\operatorname{ch}\alpha_n y)+y_n\Big(\nu\operatorname{ch}\alpha_n y+\\
&\quad+\frac{1}{2}\alpha_n y\operatorname{sh}\alpha_n y\Big)-\frac{x_n}{2}[(3-2\nu)\operatorname{sh}\alpha_n y+\alpha_n y\operatorname{ch}\alpha_n y]\Big\}\sin\alpha_n x.
\end{aligned}
\right\}
\tag{12.5}
$$

These general expressions are valid for any boundary conditions at the longitudinal plate edges $y=0$ and $y=h$.

Replacing Poisson's ratio ν by $\frac{\nu}{1+\nu}$, we obtain the general solution for plane stress under boundary conditions (12.1).

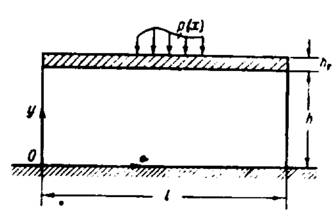


FIGURE 173.

Similar expressions can be obtained when the boundary conditions are given by (12.3).

Several examples will now be considered:

2

Let a plane punch be pressed into a rectangular plate (Figure 173) at the boundary plane $y=h$. It will be assumed that the normal plate displacements under the punch are known functions of x, being zero at the other boundary plane of the plate ($y=0$), and that the shearing stresses X vanish at $y=0$ and $y=h$. Hence, by (12.2):

at $y=h$:
$$V(x)=\sum_{n=1}^{\infty}\delta_n \sin\alpha_n x.$$

The following boundary conditions are therefore obtained:

$$\left.\begin{array}{ll} \text{at } y=0 & V_0=X_0=0; \\ \text{at } y=h & V=\sum_{n=1}^{\infty}\delta_n \sin\alpha_n x, \quad X=0. \end{array}\right\} \tag{12.6}$$

The initial functions U_0 and Y_0 are determined from the boundary conditions at $y=h$. Putting in (12.5) $v_n=x_n=0$ in accordance with (12.6), and substituting the expressions for V and X at $y=h$ in these boundary conditions, we obtain for each term of the series the following two equations with two unknowns u_n and y_n:

$$2\left[\beta_n \operatorname{ch}\beta_n-(1-2\nu)\operatorname{sh}\beta_n\right]u_n+h\left(\frac{3-4\nu}{\beta_n}\operatorname{sh}\beta_n-\operatorname{ch}\beta_n\right)y_n=$$
$$=4(1-\nu)\delta_n,$$
$$2\beta_n(\operatorname{sh}\beta_n+\beta_n\operatorname{ch}\beta_n)u_n+h\left[(1-2\nu)\operatorname{sh}\beta_n-\beta_n\operatorname{ch}\beta_n\right]y_n=0,$$

where $\beta_n=\frac{n\pi h}{l}$.

After the unknowns u_n, y_n have been determined from these equations, we can rewrite (12.5) as follows:

$$\left.\begin{aligned}
U&=-\sum_{n=1}^{\infty}\frac{\delta_n\cos\beta_n\xi}{2\Delta_n}\left\{\left[(1-2\nu)\operatorname{sh}\beta_n-\beta_n\operatorname{ch}\beta_n\right]\operatorname{ch}\beta_n\eta+\beta_n\operatorname{sh}\beta_n(\eta\operatorname{sh}\beta_n\eta)\right\},\\
V&=\sum_{n=1}^{\infty}\frac{\delta_n\sin\beta_n\xi}{2\Delta_n}\left\{\left[2(1-\nu)\operatorname{sh}\beta_n+\beta_n\operatorname{ch}\beta_n\right]\operatorname{sh}\beta_n\eta-\beta_n\operatorname{sh}\beta_n(\eta\operatorname{ch}\beta_n\eta)\right\},\\
Y&=\sum_{n=1}^{\infty}\frac{\delta_n\beta_n\sin\beta_n\xi}{h\Delta_n}\left\{(\operatorname{sh}\beta_n+\beta_n\operatorname{ch}\beta_n)\operatorname{ch}\beta_n\eta-\beta_n\operatorname{sh}\beta_n\times(\eta\operatorname{sh}\beta_n\eta)\right\},\\
X&=\sum_{n=1}^{\infty}\frac{\delta_n\beta_n^2\cos\beta_n\xi}{h\Delta_n}\left\{\operatorname{ch}\beta_n\operatorname{sh}\beta_n\eta-\operatorname{sh}\beta_n(\eta\operatorname{sh}\beta_n\eta)\right\},\\
\sigma_x&=\sum_{n=1}^{\infty}\frac{\delta_n\beta_n\sin\beta_n\xi}{h\Delta_n}\left\{(\operatorname{sh}\beta_n-\beta_n\operatorname{ch}\beta_n)\operatorname{ch}\beta_n\eta+\beta_n\operatorname{sh}\beta_n(\eta\operatorname{sh}\beta_n\eta)\right\},
\end{aligned}\right\} \tag{12.7}$$

where $\eta = \frac{y}{h}$, $\xi = \frac{x}{h}$ = dimensionless coordinates, and $\Delta_n = (1-\nu)\,\mathrm{sh}^2\beta_n$.

The magnitude $(1-\nu)$, which depends on Poisson's ratio, enters in the expressions for the stresses Y, X, σ_x only as a factor which can be written before the summation sign. If we assume that at $y=0$ and $y=h$ the displacements U and not the stresses X vanish, more complex expressions will be obtained for the stresses and displacements, and Poisson's ratio will not appear before the summation sign. In the case of boundary conditions of the mixed type (12.6) we thus obtain a peculiar generalization of M. Lévy's theorem for rectangular plates.

3

Consider as second example a double-layer plate subjected to a vertical load

$$p = \sum_{n=1}^{\infty} p_n \sin \alpha_n x.$$

We denote the elastic characteristics and thicknesses of the upper and lower layers by G_1, ν_1, h_1 and G, ν, h respectively. The directions of the coordinate axes are shown in Figure 173. It will be assumed that the upper layer behaves like a thin plate. The following boundary conditions will be assumed for the lower layer:

$$\left.\begin{aligned} &\text{at } y=0: && V_0 = X_0 = 0;\\ &\text{at } y=h: && X=0,\quad \frac{D}{G}\frac{d^4V}{dx^4} + Y = -p, \end{aligned}\right\} \tag{12.8}$$

where D = flexural rigidity of plate.

The last condition (12.8) expresses the fact that since the upper layer behaves like a thin plate, the load transmitted to the lower layer is determined from the equations of cylindrical bending of a plate. The coefficients u_n and y_n in (12.4) are found from (12.8).

We then obtain:

$$\left.\begin{aligned} U &= -\sum_{n=1}^{\infty} \frac{hp_n \cos\beta_n\xi}{2\beta_n\Delta_n}\Big\{[\beta_n \,\mathrm{ch}\,\beta_n - (1-2\nu)\,\mathrm{sh}\,\beta_n]\,\mathrm{ch}\,\beta_n\eta -\\ &\qquad - \beta_n\,\mathrm{sh}\,\beta_n\,(\eta\,\mathrm{sh}\,\beta_n\eta)\Big\},\\ V &= -\sum_{n=1}^{\infty} \frac{hp_n \sin\beta_n\xi}{2\beta_n\Delta_n}\Big\{[2(1-\nu)\,\mathrm{sh}\,\beta_n + \beta\;\mathrm{ch}\,\beta_n]\,\mathrm{sh}\,\beta_n\eta -\\ &\qquad - \beta_n\,\mathrm{sh}\,\beta_n\,(\eta\,\mathrm{ch}\,\beta_n\eta)\Big\},\\ Y &= -\sum_{n=1}^{\infty} \frac{p_n \sin\beta_n\xi}{\Delta_n}\Big\{(\mathrm{sh}\,\beta_n + \beta_n\,\mathrm{ch}\,\beta_n)\,\mathrm{ch}\,\beta_n\eta -\\ &\qquad - \beta_n\,\mathrm{sh}\,\beta_n\,(\eta\,\mathrm{sh}\,\beta_n\eta)\Big\},\\ X &= \sum_{n=1}^{\infty} \frac{p_n\beta_n \cos\beta_n\xi}{\Delta_n}\Big\{\mathrm{ch}\,\beta_n\,\mathrm{sh}\,\beta_n\eta - \mathrm{sh}\,\beta_n\,(\eta\,\mathrm{ch}\,\beta_n\eta)\Big\},\\ \sigma_x &= -\sum_{n=1}^{\infty} \frac{p_n \sin\beta_n\xi}{\Delta_n}\Big\{(\mathrm{sh}\,\beta_n - \beta_n\,\mathrm{ch}\,\beta_n)\,\mathrm{ch}\,\beta_n\eta +\\ &\qquad + \beta_n\,\mathrm{sh}\,\beta_n\,(\eta\,\mathrm{sh}\,\beta_n\eta)\Big\}, \end{aligned}\right\} \tag{12.9}$$

where

$$\Delta_n = \beta_n + \frac{1}{2}\operatorname{sh} 2\beta_n + \frac{(1-\nu) D}{Gh^3}\beta_n^3 \operatorname{sh}^2 \beta_n. \tag{12.10}$$

4

A solution in trigonometric series can also be obtained by proceeding from the fundamental differential equation of the problem instead of from (12.2) and (12.4). This will be illustrated by the above example of the bending of a thick plate.

The fundamental equation for F is in this case [cf. (11.6)]:

$$\frac{\alpha}{1-\nu}[\alpha h - \sin\alpha h \cos\alpha h] F = p(x). \tag{12.11}$$

For simplicity, only the case of a load symmetrical with respect to the y axis will be considered.

The origin of coordinates is placed at the center of the plate. We assume a solution of (12.11) for the boundary conditions (12.1) in the form:

$$F = \sum_1^\infty A_n \cos\frac{n\pi x}{2l} \qquad (n = 1, 3, 5, \ldots, (2m-1)). \tag{12.12}$$

We expand $p(x)$ in a cosine series:

$$p(x) = \sum_{n=1}^\infty p_n \cos\frac{n\pi x}{2l} \qquad (n = 1, 3, 5, \ldots, (2m-1)), \tag{12.13}$$

where:

$$p_n = \frac{4}{l}\int_0^{l/2} p(x)\cos\frac{n\pi x}{2l}\,dx.$$

Substitution of (12.12) and (12.13) in (12.11) yields:

$$A_n = -\frac{4(1-\nu)\int_0^{l/2} p(x)\cos\lambda_n x\,dx}{\frac{n\pi}{2}\left[\lambda_n h - \frac{1}{2}\sin 2\lambda_n h\right]}, \tag{12.14}$$

where

$$\lambda_n = \frac{n\pi}{2l} \qquad (n = 1, 3, 5, \ldots, (2m-1)).$$

From (11.7) and (12.12) we now obtain:

$$\left.\begin{aligned} V_0 &= \sum_{n=1}^\infty A_n\left[\operatorname{ch}\lambda_n h + \frac{\lambda_n h}{2(1-\nu)}\operatorname{sh}\lambda_n h\right]\cos\lambda_n x, \\ X_0 &= -\sum_{n=1}^\infty \frac{A_n}{1-\nu}\lambda_n^2 h \operatorname{sh}\lambda_n h \sin\lambda_n x. \end{aligned}\right\} \tag{12.15}$$

Substitution of (12.5) in Table 25 yields:

$$\left.\begin{aligned}
U &= \frac{1}{2(1-\nu)}\sum_{n=1}^{\infty} A_n[(1-2\nu)\operatorname{ch}\lambda_n h \operatorname{sh}\lambda_n y - \\
&\qquad -\lambda_n h \operatorname{sh}\lambda_n h \operatorname{sh}\lambda_n y + \lambda_n y \operatorname{ch}\lambda_n h \operatorname{ch}\lambda_n y]\sin\lambda_n x,\\
V &= \frac{1}{2(1-\nu)}\sum_{n=1}^{\infty} A_n[2(1-\nu)\operatorname{ch}\lambda_n h \operatorname{ch}\lambda_n y - \lambda_n y \operatorname{ch}\lambda_n h \operatorname{sh}\lambda_n y + \\
&\qquad + \lambda_n h \operatorname{sh}\lambda_n h \operatorname{ch}\lambda_n y]\cos\lambda_n x,\\
Y &= \sum_{n=1}^{\infty}\frac{A_n\lambda_n}{1-\nu}[\lambda_n h \operatorname{sh}\lambda_n y \operatorname{sh}\lambda_n h + \operatorname{ch}\lambda_n h \operatorname{sh}\lambda_n y - \\
&\qquad - \lambda_n y \operatorname{ch}\lambda_n h \operatorname{ch}\lambda_n y]\cos\lambda_n x,\\
X &= \sum_{n=1}^{\infty}\frac{A_n\lambda_n^2}{1-\nu}[y \operatorname{sh}\lambda_n y \operatorname{ch}\lambda_n h - h \operatorname{ch}\lambda_n y \operatorname{sh}\lambda_n h]\sin\lambda_n x,\\
\sigma_x &= \sum_{n=1}^{\infty}\frac{A_n\lambda_n}{1-\nu}[\operatorname{sh}\lambda_n y \operatorname{ch}\lambda_n h + \lambda_n y \operatorname{ch}\lambda_n y \operatorname{ch}\lambda_n h - \\
&\qquad - \lambda_n h \operatorname{sh}\lambda_n y \operatorname{sh}\lambda_n h]\cos\lambda_n x.
\end{aligned}\right\} \qquad (12.16)^*$$

5

We shall now give the exact solution in trigonometric series for a plate subjected to a load symmetrical with respect to both x and y axes (Figure 174). The solving equation is in this case:

$$\frac{\alpha}{1-\nu}\left[\alpha h + \frac{\sin 2\alpha h}{2}\right]F = -p(x), \qquad (12.17)$$

where

$$\alpha = \frac{\partial}{\partial x}.$$

We assume a solution in the form:

$$F = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{2l} \qquad (n = 1, 3, 5, \ldots, (2m-1)). \qquad (12.18)$$

Expanding $p(x)$ in a series of $\cos\frac{n\pi x}{2l}$, we obtain:

$$A_n = \frac{4(1-\nu)\int\limits_0^{l/2} p(x)\cos\lambda_n x\,dx}{\frac{n\pi}{2}\left[\lambda_n h + \frac{\sin 2\lambda_n h}{2}\right]}, \qquad (12.19)$$

where

$$\lambda_n = \frac{n\pi}{2l} \qquad (n = 1, 3, 5, \ldots, (2m-1)).$$

* [The hyperbolic functions used in this section should apparently be trigonometric functions.]

Furthermore:

$$\left.\begin{aligned} U_0 &= -\frac{1}{2(1-\nu)} \sum_{n=1}^{\infty} B_n \sin \lambda_n x, \\ Y_0 &= -\frac{1}{1-\nu} \sum_{n=1}^{\infty} C_n \cos \lambda_n x, \end{aligned}\right\} \tag{12.20}$$

where

$$\left.\begin{aligned} B_n &= A_n [(1-2\nu) \operatorname{sh} \lambda_n h - \lambda_n h \operatorname{ch} \lambda_n h], \\ C_n &= A_n \lambda_n [\operatorname{sh} \lambda_n h + \lambda_n h \operatorname{ch} \lambda_n h]. \end{aligned}\right\} \tag{12.21}$$

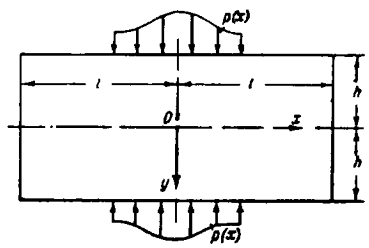


FIGURE 174.

The stresses and displacements of the plate are:

$$\left.\begin{aligned} U &= -\frac{1}{2(1-\nu)} \sum_{n=1}^{\infty} (B_n \operatorname{ch} \lambda_n y + \\ &\qquad + A_n \lambda_n y \operatorname{sh} \lambda_n h \operatorname{sh} \lambda_n y) \sin \lambda_n x, \\ V &= \sum_{n=1}^{\infty} A_n \Big[-\operatorname{sh} \lambda_n h \operatorname{sh} \lambda_n y - \frac{\lambda_n y}{2(1-\nu)} \operatorname{ch} \lambda_n h \operatorname{sh} \lambda_n y + \\ &\qquad + \frac{\lambda_n y}{2(1-\nu)} \operatorname{sh} \lambda_n h \operatorname{ch} \lambda_n y \Big] \cos \lambda_n x, \\ (1-\nu) Y &= \sum_{n=1}^{\infty} [A_n \lambda_n^2 y \operatorname{sh} \lambda_n h \operatorname{sh} \lambda_n y - C_n \operatorname{ch} \lambda_n y] \cos \lambda_n x, \\ (1-\nu) X &= \sum_{n=1}^{\infty} A_n \lambda_n^2 (h \operatorname{ch} \lambda_n h \operatorname{sh} \lambda_n y - y \operatorname{sh} \lambda_n h \operatorname{ch} \lambda_n y) \sin \lambda_n x, \\ (1-\nu) \sigma_x &= \sum_{n=1}^{\infty} A_n \lambda_n (-\operatorname{sh} \lambda_n h \operatorname{ch} \lambda_n y + \lambda_n h \operatorname{ch} \lambda_n h \operatorname{ch} \lambda_n y - \\ &\qquad - \lambda_n y \operatorname{sh} \lambda_n h \operatorname{sh} \lambda_n y) \cos \lambda_n x. \end{aligned}\right\} \tag{12.22}$$

§ 13. EXACT SOLUTION FOR A RECTANGULAR STRIP WITH ARBITRARY BOUNDARY CONDITIONS AT THE LONGITUDINAL EDGES AND HOMOGENEOUS BOUNDARY CONDITIONS AT THE LATERAL EDGES

In the preceding section, exact solutions for rectangular plates undergoing plane strain were given for the case where the boundary conditions at the lateral edges can be expressed with the aid of trigonometric series. This section will deal with the problem of finding exact solutions for a rectangular strip with arbitrary boundary conditions at $x=0$ and $x=l$ (Figure 175), and homogeneous boundary conditions at $y=0$ and $y=h$.

We first assume homogeneous boundary conditions of the mixed type, i.e., for $y=0$ and $y=h$:

$$u=\sigma_y=0. \tag{13.1}$$

This means that at the lateral edges $y=0$ and $y=h$ the strip is held by membranes rigid in their plane and flexible out of it. The initial functions U_0 and Y_0 vanish in this case.

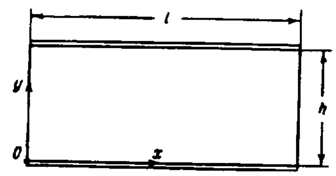


FIGURE 175.

Inserting into (13.1) the values of the operators given in Table 26, we obtain a system of two differential equations of infinitely high order in the two unknown initial functions V_0 and X_0:

$$\left.\begin{aligned}
&-[(1-\nu)\sin\alpha h+(1+\nu)\alpha h\cos\alpha h]\,V_0+\\
&\qquad+\frac{1}{2}\left[\frac{3-\nu}{\alpha}\sin\alpha h+(1+\nu)\,h\cos\alpha h\right]X_0=0,\\
&2(1+\nu)\,\alpha\,(\alpha h\cos\alpha h-\sin\alpha h)\,V_0-\\
&\qquad-[(1-\nu)\sin\alpha h+(1+\nu)\,\alpha h\cos\alpha h]\,X_0=0.
\end{aligned}\right\} \tag{13.2}$$

We introduce a function $F(x)$ satisfying the equations:

$$\left.\begin{aligned}
V_0&=-\left(\frac{1-\nu}{1+\nu}\sin\alpha h+\alpha h\cos\alpha h\right)F,\\
X_0&=-2\alpha\,(\alpha h\cos\alpha h-\sin\alpha h)\,F.
\end{aligned}\right\} \tag{13.3}$$

The second equation (13.2) is then transformed into an identity, while the first becomes:

$$(\sin^2\alpha h)\,F=0. \tag{13.4}$$

We assume a solution in the form:

$$F = Ce^{kx}. \tag{13.5}$$

Substitution of (13.5) in (13.4) leads to a transcendental equation in k:

$$\sin^2 kh = 0, \tag{13.6}$$

whose roots are:

$$k_n = \frac{n\pi}{h},$$

where n = positive integer.

The general solution of (13.4) is thus:

$$F = \sum_{n=0}^{\infty} A_n \operatorname{ch} k_n x + B_n \operatorname{sh} k_n x + C_n x \operatorname{ch} k_n x + D_n x \operatorname{sh} k_n x, \tag{13.7}$$

where A_n, B_n, C_n, D_n = arbitrary constants.

Substituting (13.7) in (13.3), we find the initial functions V_0 and X_0. Introducing these values into Table 26, we obtain the stresses and displacements of the plate when the boundary conditions are given by (13.1). For practical calculations it is more convenient to substitute first (13.3) in Table 26, simplify the results, and then use (13.7). We obtain:

$$\left.\begin{aligned}
U &= \sum_{n=1}^{\infty} (-1)^n h \Big\{k_n A_n \operatorname{ch} k_n x + k_n B_n \operatorname{sh} k_n x + \\
&\quad + \left[-\frac{1-\nu}{1+\nu} \operatorname{sh} k_n x + k_n x \operatorname{ch} k_n x\right] C_n + \\
&\quad + \left[-\frac{1-\nu}{1+\nu} \operatorname{ch} k_n x + k_n x \operatorname{sh} k_n x\right] D_n\Big\} \sin k_n y, \\
V &= \sum_{n=1}^{\infty} (-1)^n h \Big\{k_n A_n \operatorname{sh} k_n x + k_n B_n \operatorname{ch} k_n x + \\
&\quad + \left(\frac{2}{1+\nu} \operatorname{ch} k_n x + k_n x \operatorname{sh} k_n x\right) C_n + \\
&\quad + \left(\frac{2}{1+\nu} \operatorname{sh} k_n x + k_n x \operatorname{ch} k_n x\right) D_n\Big\} \cos k_n y, \\
Y &= -2 \sum_{n=1}^{\infty} (-1)^n k_n h\, k_n A_n \{\operatorname{sh} k_n x + k_n B_n \operatorname{ch} k_n x + \\
&\quad + (2 \operatorname{ch} k_n x + k_n x \operatorname{sh} k_n x) C_n + \\
&\quad + (2 \operatorname{sh} k_n x + k_n x \operatorname{ch} k_n x) D_n\} \sin k_n y, \\
X &= 2 \sum_{n=1}^{\infty} (-1)^n k_n h \{k_n A_n \operatorname{ch} k_n x + k_n B_n \operatorname{sh} k_n x + \\
&\quad + (\operatorname{sh} k_n x + k_n x \operatorname{ch} x_n k) C_n + \\
&\quad + (\operatorname{ch} k_n x + k_n x \operatorname{sh} k_n x) D_n\} \cos k_n y, \\
\sigma_x &= 2 \sum_{n=1}^{\infty} (-1)^n k_n h \{k_n A_n \operatorname{sh} k_n x + k_n B_n \operatorname{ch} k_n x + \\
&\quad + k_n x C_n \operatorname{sh} k_n x + k_n x D_n \operatorname{ch} k_n x\} \sin k_n y.
\end{aligned}\right| \tag{13.8}$$

The elementary solution in polynomials, corresponding to the zero roots of (13.6), no longer appears in the general solution (13.8) which thus represents incomplete expressions for the displacements and stresses. This elementary solution cannot be obtained in a general form by introducing the function F, since V_0 and X_0 are expressed through F by differentiations in which part of the solution in polynomials drops out.

In order to find the elementary solution, we replace the trigonometric functions in (13.2) by their expansions in infinite series. Taking only the first terms, we obtain a system of two first-order differential equations in the unknown functions V_0 and X_0:

$$-\alpha V_0 + X_0 = 0, \quad \alpha X_0 = 0. \tag{13.9}$$

It follows from (13.9) that:

$$X_0 = A_0, \quad V_0 = A_0 x + B_0. \tag{13.10}$$

Substitution of (13.10) in (10.6) [using Table 26 and taking account of (13.1)], yields:

$$U = Y = \sigma_x = 0, \quad V = A_0 x + B_0, \quad X = A_0. \tag{13.11}$$

This result corresponds to pure shear of the plate. The constant B_0 determines the rigid-body displacement of the plate in the y direction.

Adding together (13.8) and (13.11), we obtain a general solution for the displacements and stresses of the strip. To each value of n there correspond distinct states of stress and strain. The infinite set of these states forms the exact solution of the problem for boundary conditions (13.1). All individual solutions are orthogonal.

The solution obtained contains $4n+2$ constants which have to be determined from the boundary conditions at $x=0$ and $x=l$. Two boundary conditions can be formulated for each edge. Expanding the statical or geometrical magnitudes given at these edges into Fourier sine or cosine series in the interval $(0, h)$ in accordance with (13.8) and (13.11), and equating the resulting expressions to the known corresponding displacements and stresses at $x=0$ and $x=l$, we obtain for any $n \neq 0$ a system of four algebraic equations in the unknown constants A_n, B_n, C_n and D_n. When these constants are determined the problem is completely solved.

It is often advisable to introduce other constants having a clearer physical meaning. Taking $x=0$ as base plane of the strip, and assuming as before that boundary conditions (13.1) are fulfilled at $y=0$ and $y=h$, we obtain as initial functions:

$$U^* = U(0, y), \quad V^* = V(0, y), \quad X^* = X(0, y), \quad \sigma_x^* = \sigma_x(0, y),$$

which, as follows from (13.8) and (13.11), must satisfy the following relationships:

$$\left.\begin{aligned} U^* &= \sum_{n=1}^{\infty} u_n^* \sin k_n y, & V^* &= \sum_{n=0}^{\infty} v_n^* \cos k_n y, \\ X^* &= \sum_{n=0}^{\infty} x_n^* \cos k_n y, & \sigma_x^* &= \sum_{n=1}^{\infty} \sigma_n^* \sin k_n y \end{aligned}\right\} \tag{13.2}$$

Here $u_n^\bullet$, $v_n^\bullet$, $x_n^\bullet$, $\sigma_n^\bullet$ are the Fourier coefficients of the trigonometric series for the initial functions. In order to express A_n B_n, C_n and D_n throught them we set $x=0$ in (13.8) and (13.11), and equate the results obtained to the corresponding expressions (13.12). We obtain:

$$\left.\begin{aligned}
u_n^\bullet &= (-1)^n h\left(k_n A_n - \frac{1-\nu}{1+\nu} D_n\right),\\
v_n^\bullet &= (-1)^n h\left(k_n B_n + \frac{2}{1+\nu} C_n\right),\\
x_n^\bullet &= (-1)^n 2k_n h\,(k_n A_n + D_n),\\
\sigma_n^\bullet &= (-1)^n 2k_n^2 h B_n,
\end{aligned}\right\} \tag{13.13}$$

$$B_0 = v_0^\bullet \quad A_0 = x_0^\bullet. \tag{13.14}$$

It follows from (13.13) that:

$$\left.\begin{aligned}
A_n &= (-1)^n \frac{1+\nu}{4hk_n}\left(2u_n^\bullet + \frac{1-\nu}{1+\nu}\frac{x_n^\bullet}{k_n}\right),\\
B_n &= (-1)^n \frac{\sigma_n^\bullet}{2k_n^2 h},\\
C_n &= (-1)^n \frac{1+\nu}{2h}\left(v_n^\bullet - \frac{\sigma_n^\bullet}{2k_n}\right)\\
D_n &= (-1)^n \frac{1+\nu}{4h}\left(\frac{x_n^\bullet}{k_n} - 2u_n^\bullet\right)
\end{aligned}\right\} \tag{13.15}$$

Substitution of (13.14) and (13.15) in (13.8) and (13.11) yields the following general expressions for the stresses and displacements:

$$\left.\begin{aligned}
U = {}& \sum_{n=1}^{\infty}\left\{\frac{v_n^\bullet}{2}[(1+\nu)k_n x \operatorname{ch} k_n x - (1-\nu)\operatorname{sh} k_n x] + u_n^\bullet\left(\operatorname{ch} k_n x - \frac{1+\nu}{2} k_n x \operatorname{sh} k_n x\right) + \right.\\
& \left. + \frac{1+\nu}{4} x_n^\bullet x \operatorname{sh} k_n x + \frac{\sigma_n^\bullet}{4}\left[\frac{3-\nu}{k_n}\operatorname{sh} k_n x - (1+\nu) x \operatorname{ch} k_n x\right]\right\} \sin k_n y,\\
V = {}& v_0^\bullet + x_0^\bullet x + \sum_{n=1}^{\infty}\left\{v_n^\bullet\left(\operatorname{ch} k_n x + \frac{1+\nu}{2} k_n x \operatorname{sh} k_n x\right) - \right.\\
& - \frac{u_n^\bullet}{2}[(1-\nu)\operatorname{sh} k_n x + (1+\nu) k_n x \operatorname{ch} k_n x] - \frac{x_n^\bullet}{4}\left[\frac{3-\nu}{k_n} \times \right.\\
& \left.\left. \times \operatorname{sh} k_n x + (1+\nu) x \operatorname{ch} k_n x\right] - \frac{1+\nu}{4}\sigma_n^\bullet x \operatorname{sh} k_n x\right\} \cos k_n y,\\
Y = {}& \sum_{n=1}^{\infty}\left\{-(1+\nu) v_n^\bullet k_n (2\operatorname{ch} k_n x + k_n x \operatorname{sh} k_n x) + (1+\nu) u_n^\bullet k_n (\operatorname{sh} k_n x + k_n x \operatorname{ch} k_n x) - \right.\\
& \left. - \frac{x_n^\bullet}{2}[(3+\nu)\operatorname{sh} k_n x + (1+\nu) k_n x \operatorname{ch} k_n x] + \sigma_n^\bullet\left(\nu \operatorname{ch} k_n x + \frac{1+\nu}{2} k_n x \operatorname{ch} k_n x\right)\right\} \sin k_n y,\\
X = {}& x_0^\bullet + \sum_{n=1}^{\infty}\left\{(1+\nu) v_n^\bullet k_n (\operatorname{sh} k_n x + k_n x \operatorname{ch} k_n x) - (1+\nu) u_n^\bullet k_n^2 x \operatorname{sh} k_n x + \right.\\
& \left. + x_n^\bullet\left(\operatorname{ch} k_n x + \frac{1+\nu}{2} k_n x \operatorname{sh} k_n x\right) + \frac{\sigma_n^\bullet}{2}[(1-\nu)\operatorname{sh} k_n x - (1+\nu) k_n x \operatorname{ch} k_n x]\right\} \cos k_n y,\\
\sigma_x = {}& \sum_{n=1}^{\infty}\left\{(1+\nu) v_n^\bullet k_n^2 x \operatorname{sh} k_n x + (1+\nu) u_n^\bullet k_n (\operatorname{sh} k_n x - k_n x \operatorname{ch} k_n x) + \right.\\
& \left. \frac{x_n^\bullet}{2}[(1-\nu)\operatorname{sh} k_n x + (1+\nu) k_n x \operatorname{ch} k_n x] + \sigma_n^\bullet\left(-\frac{1+\nu}{2} k_n x \operatorname{sh} k_n x + \operatorname{ch} k_n x\right)\right\} \sin k_n y.
\end{aligned}\right\} \tag{13.16}$$

We can thus determine the stresses and strains of a strip in the case of plane stress with arbitrary statical, geometrical, or mixed boundary conditions at $x=0$ and $x=l$. The solution by trigonometric series (13.16) is a generalization of Filon's solution.

We could also have obtained (13.16) in a simpler way by direct substitution of (13.12) in Table 26, rewritten in terms of the variable x.

The method used to obtain (13.16) from (13.8) is a generalization of Cauchy and Krylov's method of initial parameters.

§ 14. OTHER HOMOGENEOUS BOUNDARY CONDITIONS OF THE MIXED TYPE AT THE LATERAL STRIP EDGES

Consider now a different kind of homogeneous boundary conditions of the mixed type (Figure 176). It will be assumed that at $y=0$ and $y=h$;

$$V=\tau_{xy}=0. \tag{14.1}$$

We obtain in this case:

$$V_0=X_0=0.$$

By satisfying the boundary conditions at $y=h$ we obtain, [using Table 26], a system of two differential equations of infinitely high order in the unknown initial functions U_0 and Y_0:

$$\left.\begin{aligned}&[(1-\nu)\sin\alpha h-(1+\nu)\alpha h\cos\alpha h]\,U_0+\\&\qquad+\frac{1}{2}\left[(3-\nu)\frac{\sin\alpha h}{\alpha}-(1+\nu)h\cos\alpha h\right]Y_0=0,\\&-2(1+\nu)\alpha(\sin\alpha h+\alpha h\cos\alpha h)\,U_0+\\&\qquad+[(1-\nu)\sin\alpha h-(1+\nu)\alpha h\cos\alpha h]\,Y_0=0.\end{aligned}\right\} \tag{14.2}$$

We introduce a function $F(x)$ satisfying the equations:

$$\left.\begin{aligned}U_0&=\left(\frac{1-\nu}{1+\nu}\sin\alpha h-\alpha h\cos\alpha h\right)F,\\Y_0&=2\alpha(\sin\alpha h+\alpha h\cos\alpha h)\,F,\end{aligned}\right\} \tag{14.3}$$

This transforms the second equation (14.2) into an identity. The first equation is again reduced to (13.4).

Since (13.1) and (14.1) have the same solving equations, the considerations of the preceding section apply also to this problem. As a result, we again obtain (13.7) for the solving function F.

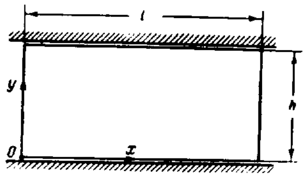


FIGURE 176

To determine the general integrals for the displacements and stresses, we have, as before, to find an elementary solution in polynomials, proceeding from the system:

$$\left.\begin{aligned} -2\nu\alpha U_0 + (1-\nu) Y_0 = 0, \\ 2(1+\nu)\alpha^2 U_0 + \nu\alpha Y_0 = 0, \end{aligned}\right\} \tag{14.4}$$

obtained from (14.2) by retaining only the first terms.

We obtain from (14.4):

$$U_0 = \frac{1-\nu}{2} B_0 x + A_0, \quad Y_0 = \nu B_0. \tag{14.5}$$

Substitution of this in (10.6) [using Table 26, and taking account of (14.1)] yields:

$$U = \frac{1-\nu}{2} B_0 x + A_0, \quad Y = \nu B_0, \quad \sigma_x = B_0. \tag{14.6}$$

Expressions (14.6) represent an elementary solution corresponding to a uniformly distributed load $\sigma_x = B_0$. The constant A_0 does not affect the states of stress and strain of the plate but determines rigid-body displacement of the strip in the x direction.

Expressions (14.5), (14.3), (13.7), and (10.6) yield:

$$\left.\begin{aligned}
U = A_0 + \frac{1-\nu}{2} B_0 x - \sum_{n=1}^{\infty} (-1)^n h \Big[k_n \operatorname{sh} k_n x A_n + \\
+ k_n \operatorname{ch} k_n x B_n + \left(\frac{2\nu}{1+\nu} \operatorname{ch} k_n x + k_n x \operatorname{sh} k_n x\right) C_n + \\
+ \left(\frac{2\nu}{1+\nu} \operatorname{sh} k_n x + k_n x \operatorname{ch} k_n x\right) D_n\Big] \cos k_n y, \\
V = \sum_{n=1}^{\infty} (-1)^n h \Big\{k_n \operatorname{ch} k_n x A_n + k_n \operatorname{sh} k_n x B_n + \\
+ \left[\frac{3+\nu}{1+\nu} \operatorname{sh} k_n x + k_n x \operatorname{ch} k_n x\right] C_n + \\
+ \left[\frac{3+\nu}{1+\nu} \operatorname{ch} k_n x + k_n x \operatorname{sh} k_n x\right] D_n\Big\} \sin k_n y, \\
Y = \nu B_0 + 2 \sum_{n=1}^{\infty} (-1)^n k_n h [k_n \operatorname{ch} k_n x A_n + k_n \operatorname{sh} k_n x B_n + \\
+ (3 \operatorname{sh} k_n x + k_n x \operatorname{ch} k_n x) C_n + \\
+ (3 \operatorname{ch} k_n x + k_n x \operatorname{sh} k_n x) D_n] \cos k_n y, \\
X = 2 \sum_{n=1}^{\infty} (-1)^n k_n h [k_n \operatorname{sh} k_n x A_n + k_n \operatorname{ch} k_n x B_n + \\
+ (2 \operatorname{ch} k_n x + k_n x \operatorname{sh} k_n x) C_n + \\
+ (2 \operatorname{sh} k_n x + k_n x \operatorname{ch} k_n x) D\,] \sin k_n y, \\
\sigma_x = B_0 - 2 \sum_{n=1}^{\infty} (-1)^n k_n h [k_n \operatorname{ch} k_n x A_n + k_n \operatorname{sh} k_n x B_n + \\
+ (\operatorname{sh} k\ x + k_n x \operatorname{ch} k_n x) C_n + \\
+ (\operatorname{ch} k_n x + k_n x \operatorname{sh} k_n x) D_n] \cos k_n y.
\end{aligned}\right\} \tag{14.7}$$

As in the preceding section, the general solution (14.7) represents an infinite set of distinct orthogonal states. Any boundary conditions for $x=0$ and $x=l$ can be satisfied by a suitable choice of the constants A_0, B_0, A_n, B_n, C_n, D_n $(n=1,\ 2,\dots,\infty)$. We can also in this case introduce constants having a clearer physical meaning by taking $x=0$ as base plane, and the magnitudes $U(0,y)=U^{\bullet}$, $V(0,y)=V^{\bullet}$, $X(0,y)=X^{\bullet}$, $\sigma_x(0,y)=\sigma_x^{\bullet}$ as initial functions.

It follows from (14.7) that when the boundary conditions are given by (14.1), we can write:

$$\left.\begin{aligned} U^{\bullet}&=\sum_{n=0}^{\infty}u_n^{\bullet}\cos k_n y, & X^{\bullet}&=\sum_{n=1}^{\infty}x_n^{\bullet}\sin k_n y,\\ V^{\bullet}&=\sum_{n=1}^{\infty}v_n^{\bullet}\sin k_n y, & \sigma_x^{\bullet}&=\sum_{n=0}^{\infty}\sigma_n^{\bullet}\cos k_n y. \end{aligned}\right\} \tag{14.8}$$

Inserting $x=0$ into (14.7) and equating the results to the corresponding equations (14.8), we obtain:

$$\left.\begin{aligned} A_n&=-(-1)^n\frac{1+\nu}{2k_nh}\left[v_n^{\bullet}+\frac{3+\nu}{2(1+\nu)}\frac{\sigma_n^{\bullet}}{k_n}\right],\\ B_n&=-(-1)^n\frac{1+\nu}{k_nh}\left[u_n^{\bullet}+\frac{\nu}{2(1+\nu)\,k_n}x_n^{\bullet}\right],\\ C_n&=(-1)^n\frac{1+\nu}{2h}\left(\frac{x_n^{\bullet}}{2k_n}+u_n^{\bullet}\right),\\ D_n&=(-1)^n\frac{1+\nu}{2h}\left(v_n^{\bullet}+\frac{1}{2}\frac{\sigma_n^{\bullet}}{k_n}\right),\\ A_0&=u_0^{\bullet},\quad B_0=\sigma_0^{\bullet}. \end{aligned}\right\} \tag{14.9}$$

Substitution of (14.9) in (14.7) then yields expressions similar to (13.16) for the displacements and stresses expressed through the initial functions (14.8):

$$\left.\begin{aligned} U={}&u_0^{\bullet}+\frac{1-\nu}{2}\sigma_0^{\bullet}x+\sum_{n=1}^{\infty}\Big\{u_n^{\bullet}\Big(\operatorname{ch}k_nx-\frac{1+\nu}{2}k\ x\operatorname{sh}k_nx\Big)+\\ &+\frac{v_n^{\bullet}}{2}[(1-\nu)\operatorname{sh}k_nx-(1+\nu)k_nx\operatorname{ch}k_nx]-\\ &-\frac{1+\nu}{4}x_n^{\bullet}x\operatorname{sh}k_nx+\\ &+\frac{\sigma_n^{\bullet}}{4}\Big[(3-\nu)\frac{\operatorname{sh}k_nx}{k_n}-(1+\nu)x\operatorname{ch}k_nx\Big]\Big\}\cos k_ny,\\ V={}&\sum_{n=1}^{\infty}\Big\{\frac{u_n^{\bullet}}{2}[(1-\nu)\operatorname{sh}k_nx+(1+\nu)k_nx\operatorname{ch}k_nx]+\\ &+v_n^{\bullet}\Big(\operatorname{ch}k_nx+\frac{1+\nu}{2}k\ x\operatorname{sh}k_nx\Big)+\\ &+\frac{x_n^{\bullet}}{4}\Big[\frac{3-\nu}{k_n}\operatorname{sh}k_nx+(1+\nu)x\operatorname{ch}k_nx\Big]+\\ &+\frac{1+\nu}{4}\sigma_n^{\bullet}x\operatorname{sh}k_nx\Big\}\sin k_ny, \end{aligned}\right\} \tag{14.10}$$

$$
\left.
\begin{aligned}
Y = {} & \nu\sigma_0^* + \sum_{n=1}^{\infty}\Big\{(1+\nu)\,u_n^* k_n(\operatorname{sh} k_n x + k_n x \operatorname{ch} k_n x) + \\
& + (1+\nu)\,v_n^* k_n(2\operatorname{ch} k_n x + k_n x \operatorname{sh} k_n x) + \\
& + \frac{x_n^*}{2}[(3+\nu)\operatorname{sh} k_n x + (1+\nu)\,k_n x \operatorname{ch} k_n x] + \\
& + \sigma_n^*\left(\nu \operatorname{ch} k_n x + \frac{1+\nu}{2}\,k_n x \operatorname{sh} k_n x\right)\Big\}\cos k_n y, \\
X = {} & \sum_{n=1}^{\infty}\Big\{(1+\nu)\,u_n^* k_n^2\, x \operatorname{sh} k_n x + (1+\nu)\,v_n^* k_n(\operatorname{sh} k_n x + \\
& + k_n x \operatorname{ch} k\ x) + x_n^*\left(\operatorname{ch} k_n x + \frac{1+\nu}{2}\,k_n x \operatorname{sh} k_n x\right) - \\
& - \frac{\sigma_n^*}{2}\,[(1-\nu)\operatorname{sh} k_n x - (1+\nu)\,k_n x \operatorname{ch} k_n x]\Big\}\sin k_n y, \\
\sigma_x = {} & \sigma_0^* + \sum_{n=1}^{\infty}\Big\{(1+\nu)\,u_n^* k_n(\operatorname{sh} k_n x - k_n x \operatorname{ch} k_n x) - \\
& - (1+\nu)\,v_n^* k_n^2\, x \operatorname{sh} k_n x - \frac{x_n^*}{2}[(1-\nu)\operatorname{sh} k_n x + \\
& + (1+\nu)\,k_n x \operatorname{ch} k_n x] + \\
& + \sigma_n^*\left(\operatorname{ch} k_n x - \frac{1+\nu}{2}\,k_n \operatorname{sh} k_n x\right)\Big\}\cos k_n y.
\end{aligned}
\right| \qquad (14.10)
$$

Equations (13.16) and (14.10) represent general solutions in trigonometric series of the two-dimensional problem of the theory of elasticity. They are generalizations of Filon's and Ribière's solutions, since the latter do not actually contain general integrals for the displacements, while (13.16) and (14.10) determine both the states of stress and strain in the strip. It is thus possible to obtain a solution for problems (13.1) and (14.1) not only when the boundary conditions for $x = 0$ and $x = l$ are statical, but also if they are geometrical or of the mixed type.

These examples do not exhaust the problems of the theory of rectangular thin plates which can be solved by the exact methods of mathematical analysis. Similar exact solutions can be obtained for homogeneous statical boundary conditions at the longitudinal edges of the strip (see /15/) or other types of boundary conditions. From these homogeneous solutions it is easy to obtain relatively simple approximations for rectangular thin plates undergoing plane strain, with arbitrary boundary conditions on all four sides of the plate. The same procedure can be applied in the presence of body forces and temperature stresses.

§ 15. THREE- AND TWO-DIMENSIONAL PROBLEMS OF THE THEORY OF THICK MULTILAYER PLATES

1

Consider a thick plate consisting of several horizontal layers having different elastic characteristics (Figure 177). Let h be the total thickness of the plate, and h_m, ν_m, G_m, be respectively the thickness and elastic constants of the m-th layer. The coordinate axes are directed as shown in Figure 177. It will be assumed that the displacement and stress vectors vary continuously at the contact plane of two layers.

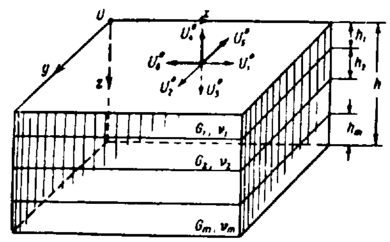


FIGURE 177.

The unknown magnitudes:

$$u, \quad v, \quad w, \quad \sigma_z, \quad \tau_{xz}, \quad \tau_{yz},$$

will be denoted as follows:

$$\left.\begin{aligned} u(x, y, z) = U_1, \quad v(x, y, z) = U_2, \quad w(x, y, z) = U_3, \\ \sigma_z(x, y, z) = U_4, \quad \tau_{xz}(x, y, z) = U_5, \quad \tau_{yz}(x, y, z) = U_6. \end{aligned}\right\} \tag{15.1}$$

The initial functions u_0, v_0, w_0, Z_0, X_0, Y_0 then become:

$$\left.\begin{aligned} u_0(x, y) = U_1^0, \quad v_0(x, y) = U_2^0 \quad w_0(x, y) = U_3^0, \\ Z_0(x, y) = U_4^0, \quad X_0(x, y) = U_5^0 \quad Y_0(x, y) = U_6^0. \end{aligned}\right\} \tag{15.2}$$

When no body forces are present, we can rewrite (2.5) as follows:

$$U_i = \sum_{k=1}^{6} L_{ik}(z)\, U_k^0 \qquad (i = 1, 2, \ldots, 6). \tag{15.3}$$

Substituting $\nu = \nu_1$, $G = G_1, z \leqslant h_1$ we determine the displacements and stresses in the first layer. These expressions contain only three of the six initial functions (15.2), since three of the latter are already known from the boundary conditions at $z = 0$. To determine the unknown magnitudes of the second layer we first obtain the displacements and stresses at the contact plane of the first and second layers at $z = h_1$, which form the initial functions for the second layer. Then, substituting in (15.3) $\nu = \nu_2$, $G = G_2$, and the expressions for the initial functions of the second layer, we determine the displacements and stresses in the second layer:

$$U_1 = \sum_{k=1}^{6} L_{ik}^{*}(z)\, U_k^0 \qquad (i = 1, 2, \ldots, 6), \tag{15.4}$$

where:

$$L^{*}_{ik}(z)=\sum_{j=1}^{6} L^{(2)}_{ij}(z)\,L^{(1)}_{jk}(h_1) \qquad (h_1 \leqslant z \leqslant h_2). \tag{15.5}$$

We denote by $L^{(1)}_{jk}(h_1)$ and $L^{(2)}_{ij}(z)$ respectively, the operators L_{ik} in (15.3) corresponding to the first layer at $z=h_1$ and to the second layer at any arbitrary value of z:

$$\left.\begin{aligned} L^{(1)}_{jk}(h_1) &= L_{jk}(\nu_1, G_1, h_1), \\ L^{(2)}_{ij}(z) &= L_{ij}(\nu_2, G_2, z) \qquad (h_1 \leqslant z \leqslant h_2). \end{aligned}\right\} \tag{15.6}$$

The matrix $\|L^{*}_{ik}(z)\|$ is thus the product of the matrix $\|L^{(2)}_{ij}(z)\|$ and the matrix $\|L^{(1)}_{jk}(h_1)\|$, and is therefore a function of the three initial functions corresponding to $z=0$. Similarly, the matrix $\|L_{ik}(z)\|$ for the displacements and stresses in the m-th layer of the plate is the product of the matrices:

$$\|L^{(j)}_{ik}(h_j)\| \ (j=1, 2, \ldots, m-1) \text{ and } \|L^{(m)}_{ik}(z)\|\,(h_{m-1} \leqslant z \leqslant h_m).$$

Determining in this way the displacements and stresses at the bottom $z=h$ of the plate, and inserting the boundary conditions for this plane, we obtain the system of differential equations of the three-dimensional problem considered, from which the three unknown initial functions can be obtained, and thus the states of strain and stress of the multilayer plate determined.

2

Consider the case of plane strain of a multilayer plate (Figure 178). We introduce the following symbols:

$$\left.\begin{aligned} &u(x, y) = U_1, \quad v(x, y) = U_2, \\ &\sigma_y(x, y) = U_3, \quad \tau_{xy}(x, y) = U_4, \quad \sigma_x(x, y) = U_5, \end{aligned}\right\} \tag{15.7}$$

$$\left.\begin{aligned} &u_0(x) = U^0_1, \quad v_0(x) = U^0_2, \\ &Y_0(x) = U^0_3, \quad X_0(x) = U^0_4. \end{aligned}\right\} \tag{15.8}$$

We can then rewrite (10.6) as follows:

$$U_i = \sum_{k=1}^{4} L_{ik}(y)\,U^0_k \qquad (i=1, 2, 3, 4). \tag{15.9}$$

The displacements and stresses in the m-th layer of the plate are:

$$U_i = \sum_{k=1}^{4} L^{*}_{ik}(y)\,U^0_k \qquad (i=1, 2, 3, 4), \tag{15.10}$$

where the matrix $\| L^{*}_{ik}(y) \|$ is the product of the matrices:

$$\| L^{(j)}_{ik}(h_j) \| \qquad (j = 1, 2, \ldots, m-1) \text{ and } \| L^{(m)}_{ik}(y) \| \qquad (h_{m-1} \leqslant y \leqslant h_m).$$

and contains only two of the four initial functions. The other two initial functions are determined directly from the boundary conditions at $y = 0$.

Inserting (15.9) into the boundary conditions at $y = h$ yields a system of two ordinary differential equations of infinitely high order with constant coefficients whose solution determines the two remaining initial functions.

We shall consider now in more detail problems with boundary conditions for the longitudinal plate edges $x = 0$ and $x = l$. Let the initial functions be represented by the following series with constant coefficients:

$$\left.\begin{aligned} U_1^0 &= \sum_{n=1}^{\infty} u_{1n}^0 \sin \alpha_n x, & U_3^0 &= \sum_{n=0}^{\infty} u_{3n}^0 \cos \alpha_n x, \\ U_2^0 &= \sum_{n=0}^{\infty} u_{2n}^0 \cos \alpha_n x, & U_4^0 &= \sum_{n=1}^{\infty} u_{4n}^0 \sin \alpha_n x, \end{aligned}\right\} \tag{15.11}$$

where

$$\alpha_n = \frac{n\pi}{l}.$$

The boundary conditions at the plate edges $x = 0$ and $x = l$ are:

$$U_1 = U_4 = 0 \qquad (u = \tau_{xy} = 0). \tag{15.12}$$

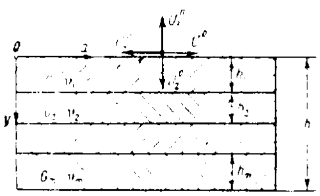


FIGURE 178.

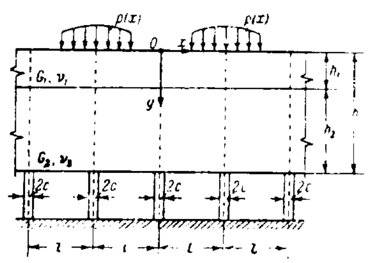


FIGURE 179.

These conditions are satisfied for each span of a multispan plate resting on an infinite number of identical and equidistant supports and subjected to a load symmetrical with respect to the ends of each span (Figure 179).

As already mentioned (see section 12), the representation of the initial functions in the form (15.11) corresponds to Ribière's solution. Interchanging the sines and cosines yields Filon's solution which corresponds to the following boundary conditions:

$$U_2 = U_3 = 0. \tag{15.13}$$

Substituting (15.11) in (15.9) and inserting the values of the operators L_{ik} given in Table 25, we obtain:

$$
\left.
\begin{aligned}
U_1 = {} & \sum_{n=1}^{\infty} \Big\{ u_{1n}^0 \Big[\operatorname{ch} \alpha_n y + \frac{1}{2(1-\nu)} \alpha_n y \operatorname{sh} \alpha_n y \Big] + \frac{u_{2n}^0}{2(1-\nu)} \times \\
& \times [(1-2\nu) \operatorname{sh} \alpha_n y + \alpha_n y \operatorname{ch} \alpha_n y] + \frac{u_{3n}^0}{4G(1-\nu)} y \operatorname{sh} \alpha_n y + \\
& + \frac{u_{4n}^0}{4G(1-\nu)} \Big[\frac{3-4\nu}{\alpha_n} \operatorname{sh} \alpha_n y + y \operatorname{ch} \alpha_n y \Big] \Big\} \sin \alpha_n x, \\
U_2 = {} & u_{20}^0 = \frac{u_{30}^0 (1-2\nu)}{2(1-\nu)} y + \\
& + \sum_{n=1}^{\infty} \Big\{ \frac{u_{1n}^0}{2(1-\nu)} [(1-2\nu) \operatorname{sh} \alpha_n y - \alpha_n y \operatorname{ch} \alpha_n y] + \\
& + u_{2n}^0 \Big[\operatorname{ch} \alpha_n y - \frac{1}{2(1-\nu)} \alpha_n y \operatorname{sh} \alpha_n y \Big] + \frac{u_{3n}^0}{4G(1-\nu)} \times \\
& \times \Big(\frac{3-4\nu}{\alpha_n} \operatorname{sh} \alpha_n y - y \operatorname{ch} \alpha_n y \Big) - \frac{u_{4n}^0}{4G(1-\nu)} y \operatorname{sh} \alpha_n y \Big\} \cos \alpha_n x, \\
U_3 = {} & u_{30}^0 + \sum_{n=1}^{\infty} \Big\{ -\frac{u_{1n}^0 G}{1-\nu} \alpha_n^2 y \operatorname{sh} \alpha_n y + \\
& + \frac{u_{2n}^0 G}{1-\nu} \alpha_n (\operatorname{sh} \alpha_n y - \alpha_n y \operatorname{ch} \alpha_n y) + \\
& + u_{3n}^0 \Big(\operatorname{ch} \alpha_n y - \frac{1}{2(1-\nu)} \alpha_n y \operatorname{sh} \alpha_n y \Big) - \\
& - \frac{u_{4n}^0}{2(1-\nu)} \Big[(1-2\nu) \operatorname{sh} \alpha_n y + \alpha_n y \operatorname{ch} \alpha_n y \Big] \Big\} \cos \alpha_n x, \\
U_4 = {} & \sum_{n=1}^{\infty} \Big\{ \frac{u_{1n}^0 G}{1-\nu} \alpha_n (\operatorname{sh} \alpha_n y + \alpha_n y \operatorname{ch} \alpha_n y) + \frac{u_{2n}^0 G}{1-\nu} \alpha_n^2 y \operatorname{sh} \alpha_n y - \\
& - \frac{u_{3n}^0}{2(1-\nu)} [(1-2\nu) \operatorname{sh} \alpha_n y - \alpha_n y \operatorname{ch} \alpha_n y] + \\
& + u_{4n}^0 \Big[\operatorname{ch} \alpha_n y + \frac{1}{2(1-\nu)} \alpha_n y \operatorname{sh} \alpha_n y \Big] \Big\} \sin \alpha_n x, \\
U_5 = {} & \frac{u_{30}^0 \nu}{1-\nu} + \sum_{n=1}^{\infty} \Big\{ \frac{u_{1n}^0 G}{1-\nu} \alpha_n (2 \operatorname{ch} \alpha_n y + \alpha_n y \operatorname{sh} \alpha_n y) + \\
& + \frac{u_{2n}^0 G}{1-\nu} \alpha_n (\operatorname{sh} \alpha_n y + \alpha_n y \operatorname{ch} \alpha_n y) + \\
& + \frac{u_{3n}^0}{1-\nu} \Big(\nu \operatorname{ch} \alpha_n y + \frac{1}{2} \alpha_n y \operatorname{sh} \alpha_n y \Big) + \\
& + \frac{u_{4n}^0}{2(1-\nu)} [(3-2\nu) \operatorname{sh} \alpha_n y + \alpha_n y \operatorname{ch} \alpha_n y] \Big\} \cos \alpha_n x.
\end{aligned}
\right| \qquad (15.14)
$$

Putting $\nu = \nu_1$, $G = G_1$, we obtain from these expressions the stresses and strains in the first layer of the plate. The coefficients u_{in}^0 ($i = 1, 2, 3, 4$) represent unknown magnitudes determined from the boundary conditions at $y = 0$ and $y = h$.

Substituting (15.11) in (15.10), we obtain the displacements and stresses in the m-th layer. These can also be determined by a different and simpler procedure. Each term of any series (15.11) is orthogonal in the interval $(0, l)$ to all other terms of the same series. Expressions (15.14) can therefore be considered as an infinite set of independent and orthogonal states of stress and strain. The displacements and stresses in the m-th layer are therefore, in accordance with (15.14):

$$u_{in} = \sum_{k=1}^{4} a_{ik}^{(n)*}(y)\, u_{kn}^{0} \qquad (i = 1, 2, \ldots, 5), \tag{15.15}$$

where the matrix $\|a_{ik}^{(n)*}(y)\|$ is the product of the matrices:

$$\|a_{ik}^{(n)}(\nu_j, G_j, h_j)\| \quad (j = 1, 2, \ldots, m-1) \text{ and } \|a_{ik}^{(n)}(\nu_m, G_m, y)\|.$$

Consider as example the equilibrium of a double-layer plate subjected to a vertical uniformly distributed load p, acting on the upper surface of the plate (Figure 179). We denote by $2c$ the width of the plate supports, and by l the distance between the support centers; the coordinates are directed as shown.

It will be assumed that due to the external load only normal stresses U_3, distributed uniformly over the plate width, arise in the supports. Thus, for $y=h$, the stresses U_3 are constant = $\left(-\frac{1}{2}\frac{pl}{c}\right)$ at the supports and zero between them.

The boundary conditions at the upper plane $y=0$ are:

$$U_3^0 = u_{30}^0 = -p, \quad U_4^0 = 0.$$

The normal load acting on the lower plane of the plate can be represented as a Fourier series in the interval $(0, l)$:

$$U_3(x, h) = -p\Big(1 + \sum_{n=2,4,6}^{\infty} \frac{2}{c\alpha_n} \sin\alpha_n c \cos\alpha_n x\Big). \tag{15.16}$$

The first term of this series corresponds to the load $U_{30}(h) = -p$. The combined action of this load and the load $U_{30}^0 = -p$ causes a uniform compression of the plate. The loads represented by the remaining terms of the series are statically equivalent to zero in the interval $(0, l)$.

Substituting (15.15) and (15.16) in the boundary conditions for $y = h$, we obtain:

$$\left.\begin{aligned} a_{31}^{(n)*}u_{1n}^0 + a_{32}^{(n)*}u_{2n}^0 &= -\frac{2p}{c\alpha_n}\sin\alpha_n c, \\ a_{41}^{(n)*}u_{1n}^0 + a_{42}^{(n)*}u_{2n}^0 &= 0, \end{aligned}\right\} \tag{15.17}$$

where:

$$a_{ik}^{(n)*} = \sum_{j=1}^{4} a_{ij}^{(n)}(\nu_2, G_2, h_2)\, a_{jk}^{(n)}(\nu_1, G_1, h_1). \tag{15.18}$$

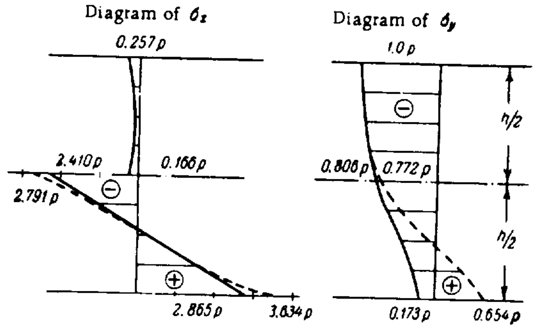


FIGURE 180.

All the unknown coefficients u_{1n}^0 and u_{2n}^0 $(n = 2, 4, 6, \ldots)$ can be obtained from (15.17). It is then possible to determine the stresses and strains in the plate.

The normal stresses $U_3 = \sigma_y$ and $U_5 = \sigma_x$ in the middle section $x = \frac{1}{2}l$ of the span are given in Figure 180. The plate considered has the following dimensions and elastic characteristics:

$$h_1 = h_2 = \frac{3}{2}l, \quad G_2 = 10\,G_1, \quad \nu_1 = \nu_2 = 0.3.$$

Two curves have been plotted in each graph; the full line represents the sum of three terms of series (15.16), while the broken line represents two terms of the series. It is seen from the diagrams that the normal stress σ_x has a discontinuity at the contact plane of the layers.

It was assumed that there are no relative displacements between the points of the lower and the upper layer at the contact plane. We shall now assume that the contact plane is perfectly smooth, so that the shearing stresses $U_4 = \tau_{xy}$ vanish there. The displacements $U_1 = u$ are discontinuous, while the displacements $U_2 = v$ and normal stresses $U_3 = \sigma_y$ are continuous across this plane.

Representing the initial function of the lower layer by the series:

$$U_1^{0'}(x) = \sum_{n=2,4,6}^{\infty} u_{1n}^{0'} \sin \alpha_n x, \tag{15.19}$$

we obtain from the boundary conditions:

$$\left.\begin{aligned} &a_{41}^{(n)}(\nu_1, G_1, h_1)\, u_{1n}^0 + a_{42}^{(n)}(\nu_1, G_1, h_1)\, u_{2n}^0 = 0,\\ &a_{31}^{(n)*} u_{1n}^0 + a_{32}^{(n)*} u_{2n}^0 + a_{31}^{(n)}(\nu_1, G_2, h_2)\, u_{1n}^{0'} = -\frac{2p}{c\alpha_n}\sin \alpha_n c,\\ &a_{41}^{(n)*} u_{1n}^0 + a_{42}^{(n)*} u_{2n}^0 + a_{41}^{(n)}(\nu_2, G_2, h_2)\, u_{1n}^0 = 0, \end{aligned}\right\} \tag{15.20}$$

where the coefficients $a_{ik}^{(n)*}$ are determined by (15.18).

The first and third equations (15.20) state that U_4 vanishes at the contact plane of the layers and at the bottom of the plate. The second equation expresses the equilibrium condition with respect to U_3 at the plate bottom. These equations are sufficient for determining the stresses and strains in the plate.

The stresses $U_3=\sigma_y$ and $U_6=\sigma_x$ in the middle section of the plate have been plotted in Figure 181. The dimensions and elastic characteristics are the same as in the preceding example. The diagrams represent the sums of two terms of (15.16). The distributions of the normal stresses over the middle section are practically linear. The stresses σ_x have a discontinuity at the contact plane of the layers.

3.

A solution by trigonometric series is also possible in the three-dimensional problem, provided one of the following conditions is fulfilled at the longitudinal edges of the plate: 1) the shearing stresses in, and the displacements normal to, the boundary plane vanish or, 2) the normal stresses in, and the tangential displacements of, the boundary plane vanish in the two-dimensional problem. The boundary conditions of the first kind correspond to (15.12), and those of the second kind, to (15.13).

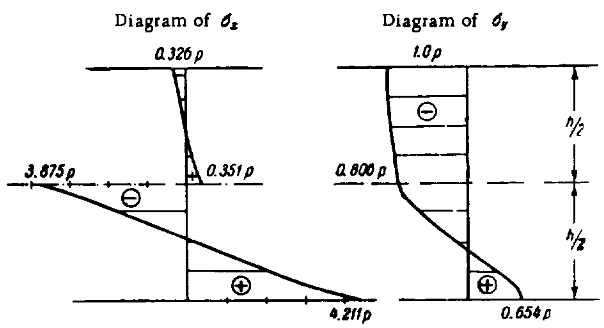


FIGURE 181.

When boundary conditions of the first type obtain on all sides of the plate ($x=0,\ x=a,\ y=0,\ y=b$), the initial functions can be represented in the form:

$$\left.\begin{aligned}
U_1^0&=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}u_{1nm}^0\sin\alpha_n x\cos\beta_m y, & U_4^0&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}u_{4nm}^0\cos\alpha_n x\cos\beta_m y,\\
U_2^0&=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}u_{2nm}^0\cos\alpha_n x\sin\beta_m y, & U_5^0&=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}u_{5nm}^0\cos\alpha_n x\sin\beta_m y,\\
U_3^0&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}u_{3nm}^0\cos\alpha_n x\cos\beta_m y, & U_6^0&=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}u_{6nm}^0\sin\alpha_n x\cos\beta_m y,
\end{aligned}\right\} \tag{15.21}$$

where

$$\alpha_n=\frac{n\pi}{a},\quad \beta_m=\frac{m\pi}{b}.$$

If boundary conditions of the second type are fulfilled on all sides, it is necessary to interchange sines and cosines in (15.21). If conditions of the first type are fulfilled on two opposite sides, and conditions of the second type on the remaining sides, the trigonometric functions of one argument are unchanged, while in those of the other sines and cosines have to be interchanged.

In the three-dimensional case, the stresses and strains in a multilayer plate are determined in the same way as in the two-dimensional case. The unknown coefficients in (15.21) are found from the boundary conditions at $z=0$ and $z=h$.

The stresses and strains in a plate, corresponding to (15.21) can be represented as follows. Consider an [infinite] multilayer plate supported by a large number of rows of columns (Figure 182) arranged in two orthogonal directions. The distances between the centers of adjacent columns are uniform, being a in one direction and b in the other.

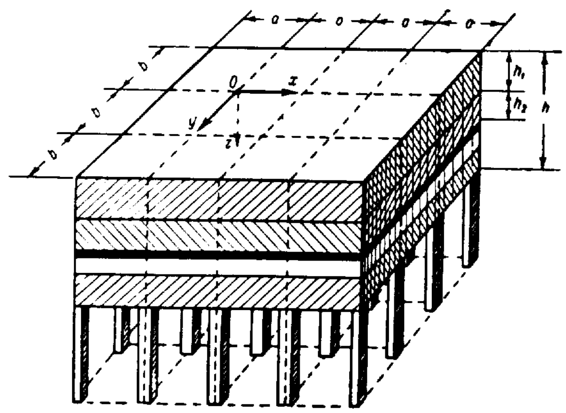


FIGURE 182.

The planes passing through the centers of the columns in the direction of the rows form two families of orthogonal planes of symmetry. If an external load, symmetrical with respect to all these planes, acts on the plate, all plate elements which form rectangular plates supported on four adjacent columns, will be under the same conditions. Boundary conditions of the first type will be fulfilled on all sides of each plate; the initial functions determining the stresses and strains in the plate are then given by (15.21). If the load acting on the plate is antisymmetrical with respect to all planes of both families, boundary conditions of the second type will be fulfilled on all sides of each plate element; the initial functions are then obtained from (15.21) by interchanging the sines and cosines. If, finally, the load acting on the plate is symmetrical with respect to all planes of one family and antisymmetrical with respect to the other, the problem will be of the mixed type: the functions of one argument in (15.21) remain unchanged while sines and cosines are interchanged in the functions of the other argument. The general case of a continuous plate subjected to an arbitrary external load can be considered as a combination of the above-mentioned symmetrical and antisymmetrical loads.

§ 16. ELASTIC MULTILAYER FOUNDATION

The above theory of thick multilayer plates can also be applied to determine the stresses and strains appearing in an elastic multilayer foundation when an external load is applied to its surface (Figure 183).

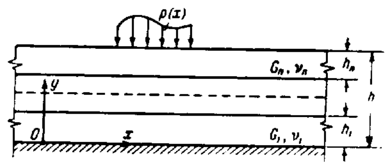


FIGURE 183.

The displacements and stresses in the first layer, which is in a state of plane strain, can be represented in the form:

$$U_i = \sum_{k=1}^{4} \int_{-\infty}^{\infty} [A_{ik}(y, \alpha) f_i(x, \alpha) u_k^0(\alpha) + B_{ik}(y, \alpha) g_i(x, \alpha) u_k^{0*}(\alpha)]\, d\alpha \tag{16.1}$$

$$(i = 1, 2, \ldots, 5),$$

where $f_i(x, \alpha) = \sin \alpha x$ for $i = 1, 4$; $f_i(x, \alpha) = \cos \alpha x$, for $i = 2, 3, 5$; $g_i(x, \alpha) = \cos \alpha x$ for $i = 1, 4$; $g_i(x, \alpha) = \sin \alpha x$ for $i = 2, 3, 5$; the functions A_{ik} are those entering in (15.14):

$$A_{11} = \operatorname{ch} \alpha y + \frac{\alpha y \operatorname{sh} \alpha y}{2(1-\nu_1)}, \qquad A_{23} = \frac{1}{4G(1-\nu_1)}\left(\frac{3-4\nu_1}{\alpha} \operatorname{sh} \alpha y - y \operatorname{ch} \alpha y\right)$$

etc.; the functions B_{ik} are determined from expressions similar to (15.14) but corresponding to a different representation of the initial functions (Filon's form).

When the displacements and stresses across the contact planes of the layers are continuous, the stresses and displacements in the m-th layer of an infinite plate are:

$$U_i = \sum_{k=1}^{4} \int_{-\infty}^{\infty} [A_{ik}^{*} f_i u_k^0 + B_{ik}^{*} g_i u_k^{0*}]\, d\alpha, \tag{16.2}$$

where the matrices $\|A_{ik}^{*}\|$ and $\|B_{ik}^{*}\|$ are the products of the matrices:

$$\| A_{ik}^{(j)}(\alpha, h_j, \nu_j, G_j)\| \text{ and } \| A_{ik}^{(m)}(\alpha, y, \nu_m, G_m)\|,$$

$$\|B_{ik}^{(j)}(\alpha, h_j, \nu_j, G_j)\| \text{and} \|B_{ik}^{(m)}(\alpha, y, \nu_m, G_m)\|$$

$$(j = 1, 2, \ldots, m-1), \qquad (h_{m-1} \leqslant y \leqslant h_m).$$

respectively.

The unknown functions u_k^0 and u_k^{0*} have to be found from the boundary conditions at $y=0$ and $y=h$.

Consider as example an infinite elastic foundation, on a bounded region of which acts an external load (Figure 183). It will be assumed that this foundation lies on a rigid subsoil, and that there is no friction between the foundation and the subsoil. In this case the vertical displacements U_2 and the shearing stresses U_4 vanish for $y=0$, so that in (16.2) we must put: $u_2^0=u_2^{0*}=u_4^0=u_4^{0*}=0$. To determine the unknown functions u_1^0, u_1^{0*}, u_3^0, u_3^{0*}, we shall use the statical boundary conditions for $y=h$. Let only a normal distributed load $p(x)$, differing from zero in the interval $a_1 \leqslant x \leqslant a_2$, act at $y=h$ (Figure 183). We represent $p(x)$ as a Fourier integral:

$$p(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha \int_{a_1}^{a_2} p(\lambda)\cos\alpha(\lambda-x)\,d\lambda. \tag{16.3}$$

Equating the expressions for the stresses U_3 and U_4, obtained from (16.2) for $y=h$, to the boundary value (16.3) and to zero respectively, we obtain:

$$\left.\begin{aligned}
&\sum_{i=1,3} A_{31}^{*}(\alpha h)\,u_i^0=-\frac{1}{2\pi}\int_{a_1}^{a_2} p(\lambda)\cos\alpha\lambda\,d\lambda; \quad \sum_{i=1,3} A_{4i}^{*}(\alpha h)\,u_i^0=0;\\
&\sum_{i=1,3} B_{31}^{*}(\alpha h)\,u_i^{0*}=-\frac{1}{2\pi}\int_{a_1}^{a_2} p(\lambda)\sin\alpha\lambda\,d\lambda; \quad \sum_{i=1,3} B_{4i}^{*}(\alpha h)\,u_i^{0*}=0.
\end{aligned}\right\} \tag{16.4}$$

Having found u_i^0 and u_i^{0*} $(i=1,3)$, from (16.4) we determine the stresses and strains in the plate from (16.2). The expressions for the displacements and stresses cannot be obtained in finite form, since the integrals (16.2) cannot be expressed in elementary functions and must be evaluated numerically.

The three-dimensional equilibrium problem of a multilayer foundation extending to infinity in two directions can be similarly treated. We proceed in this case from the sum of the four different representations of the initial functions. One representation is (15.21), while the others are obtained from (15.21) by suitably interchanging sines and cosines. In order to satisfy the boundary conditions for $z=h$ it is necessary to use a double Fourier integral.

TABLE 1

Function $\Phi_1(z) = \operatorname{sh} z \cos \gamma z$, where $z = \bar{\alpha}\eta$

z \ γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0	0	0	0	0	0	0	0	0
0.05	0.05000	0.05000	0.04999	0.04999	0.04998	0.04998	0.04997	0.04996	0.04995
0.10	0.10019	0.10018	0.10015	0.10012	0.10007	0.10002	0.09995	0.09988	0.09979
0.15	0.15058	0.15053	0.15045	0.15033	0.15017	0.14999	0.14977	0.14952	0.14923
0.20	0.20130	0.20118	0.20098	0.20070	0.20033	0.19989	0.19937	0.19877	0.19809
0.25	0.25253	0.25229	0.25190	0.25135	0.25063	0.24977	0.24875	0.24757	0.24624
0.30	0.30438	0.30397	0.30329	0.30233	0.30110	0.29960	0.29783	0.29579	0.29349
0.35	0.35697	0.35631	0.35522	0.35370	0.35173	0.34934	0.34652	0.34328	0.33961
0.40	0.41042	0.40943	0.40780	0.40550	0.40256	0.39898	0.39475	0.38990	0.38443
0.45	0.46486	0.46345	0.46110	0.45782	0.45360	0.44848	0.44243	0.43551	0.45096
0.50	0.52045	0.51849	0.51525	0.51071	0.50490	0.49782	0.48950	0.47996	0.46922
0.55	0.57727	0.57466	0.57029	0.56422	0.55642	0.54695	0.53582	0.52308	0.50875
0.60	0.63550	0.63207	0.62636	0.61840	0.60822	0.59584	0.58132	0.56470	0.54606
0.65	0.69527	0.69087	0.68354	0.67333	0.66027	0.64443	0.62586	0.60465	0.58088
0.70	0.75672	0.75116	0.74191	0.72904	0.71259	0.69265	0.66932	0.64271	0.61295
0.75	0.82000	0.81308	0.80158	0.78559	0.76517	0.74046	0.71157	0.67869	0.64198
0.80	0.88527	0.87677	0.87154	0.84333	0.81800	0.78774	0.75246	0.71235	0.66769
0.85	0.95266	0.94233	0.92519	0.89182	0.87105	0.83444	0.79180	0.74345	0.68972
0.90	1.02236	1.00993	0.98933	0.96072	0.92433	0.88046	0.82946	0.77175	0.70778
0.95	1.09451	1.07969	1.05512	1.02104	0.97774	0.92565	0.86518	0.79695	0.72149
1.00	1.16933	1.16353	1.12271	1.08243	1.03133	0.96994	0.89884	0.81877	0.73052
1.05	1.24694	1.22631	1.19214	1.14489	1.08812	1.01190	0.93014	0.83690	0.73445
1.10	1.32758	1.30346	1.26358	1.20843	1.13867	1.05515	0.95888	0.85101	0.73326
1.15	1.41139	1.38337	1.33704	1.27309	1.19229	1.09578	0.98474	0.86074	0.72534
1.20	1.49861	1.46620	1.41270	1.33887	1.24582	1.13483	1.00750	0.86570	0.71145
1.25	1.58941	1.55212	1.49059	1.40581	1.29909	1.17211	1.02601	0.86552	0.69070
1.30	1.68404	1.64131	1.57085	1.47389	1.35205	1.20739	1.04238	0.85975	0.66262
1.35	1.78270	1.73391	1.65538	1.54310	1.40455	1.24047	1.05380	0.84796	0.62669
1.40	1.88568	1.83015	1.73880	1.61344	1.45648	1.27104	1.06073	0.82966	0.58237
1.45	1.99310	1.93015	1.82666	1.68486	1.50766	1.29886	1.06277	0.80438	0.52911
1.50	2.10537	2.03419	1.91731	1.75738	1.55795	1.32858	1.05947	0.77157	0.46633
1.55	2.22261	2.09738	2.01072	1.83091	1.60714	1.34488	1.05036	0.73067	0.39343
1.60	2.34721	2.25688	2.10888	1.90705	1.65648	1.36358	1.03586	0.68170	0.31008
1.65	2.47338	2.37216	2.20646	1.98087	1.70144	1.37582	1.01281	0.62230	0.21486
1.70	2.60748	2.49417	2.30895	2.05716	1.74606	1.38464	0.98327	0.55357	0.10791
1.75	2.74774	2.62123	2.40064	2.13422	1.78862	1.38834	0.94578	0.47429	−0.01172
1.80	2.89286	2.75358	2.52353	2.21195	1.82888	1.38673	0.89977	0.38372	−0.14470
1.85	3.04832	2.89142	2.63575	2.29021	1.86642	1.37902	0.84451	0.28119	−0.29171
1.90	3.20933	3.03501	2.75146	2.35889	1.90102	1.36475	0.77939	0.16592	−0.45346
1.95	3.37793	3.18466	2.87060	2.44781	1.93219	1.34337	0.70358	0.03719	−0.63066
2.00	3.55458	3.34055	2.99339	2.52687	1.95959	1.31423	0.61646	−0.10590	−0.82402
2.05	3.73956	3.50301	3.10058	2.60579	1.98274	1.27666	0.51709	−0.26412	−1.03427
2.10	3.93350	3.67232	3.24978	2.68443	2.00116	1.22996	0.40472	−0.43834	−1.26210
2.15	4.13668	3.84875	3.38350	2.76255	2.01429	1.17338	0.27840	−0.62941	−1.50822
2.20	4.34969	4.03257	3.52107	2.83985	2.02174	1.10616	0.13723	−0.83829	−1.77339
2.25	4.57290	4.22416	3.66240	2.91607	2.02269	1.02741	−0.01970	−1.06583	−2.05820
2.30	4.80697	4.42376	3.80763	2.99091	2.01670	0.93624	−0.19348	−1.31303	−2.36347
2.35	5.05016	4.63179	3.95659	3.06402	2.00292	0.83173	−0.38516	−1.58087	−2.68966
2.40	5.30957	4.84849	4.10957	3.13499	1.98074	0.71290	−0.59576	−1.87027	−3.03758
2.45	5.57919	5.07431	4.26623	3.20344	1.94925	0.57873	−0.82642	−2.18223	−3.40766
2.50	5.86210	5.30953	4.42687	3.26892	1.90775	0.42799	−1.07845	−2.51779	−3.80055
2.55	6.15862	5.55456	4.59123	3.33099	1.85525	0.25961	−1.35284	−2.87790	−4.21674
2.60	6.46972	5.80982	4.75935	3.38901	1.79084	0.07230	−1.65112	−3.26355	−4.65659
2.65	6.79584	6.07564	4.93115	3.44247	1.71345	−0.13520	−1.97442	−3.67583	−5.12050
2.70	7.13793	6.35242	5.10662	3.49079	1.62204	−0.36424	−2.32416	−4.11566	−5.60891
2.75	7.49655	6.64058	5.28546	3.53325	1.51534	−0.61629	−2.70189	−4.58403	−6.12180
2.80	7.87293	6.94069	5.46778	3.56906	1.39238	−0.81092	−3.10842	−5.08194	−6.65962
2.85	8.26730	7.25289	5.65319	3.59750	1.25157	−1.19532	−3.54581	−5.61029	−7.22196

z \ γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2.90	8.68123	7.57796	5.84188	3.61784	1.09168	-1.52563	-4.01547	-6.17010	-7.80934
2.95	9.11516	7.91621	6.03314	3.62895	0.91124	-1.88526	-4.51876	-6.76203	-8.42094
3.00	9.57047	8.26815	6.22721	3.63007	0.70866	-2.27606	-5.05752	-7.38707	-9.05686
3.05	10.04774	8.63418	6.42342	3.62000	0.48225	-2.69997	-5.63294	-8.04607	-9.71624
3.10	10.54843	9.01490	6.62183	3.59763	0.23028	-3.15889	-6.24701	-8.73943	-10.39890
3.15	11.07336	9.41081	6.81606	3.56177	-0.04892	-3.65482	-6.90096	-9.46811	-11.10341
3.20	11.62428	9.82242	7.02326	3.51114	-0.35758	-4.18993	-7.59685	-10.23253	-11.82927
3.25	12.20160	10.25015	7.22537	3.44427	-0.69761	-4.76636	-8.33604	-11.03313	-12.57552
3.30	12.80738	10.69479	7.42810	3.35983	-1.07112	-5.36645	-9.12047	-11.87028	-13.33901
3.35	13.44242	11.15675	7.36047	3.25599	-1.48060	-6.05250	-9.95143	-12.74411	-14.11966
3.40	14.10859	11.63661	7.83242	3.13135	-1.92814	-6.76704	-10.83103	-13.65499	-14.91552
3.45	14.80615	12.13509	8.03268	2.98386	-2.41663	-7.53249	-11.76030	-14.60255	-15.72362
3.50	15.53965	12.65246	8.23112	2.81175	-2.94872	-8.35155	-12.74163	-15.58680	-16.54197
3.55	16.30772	13.18962	8.42657	2.61284	-3.52681	-9.22679	-13.77574	-16.60721	-17.36725
3.60	17.11336	13.74719	8.61848	2.38479	-4.15446	-10.16123	-14.86516	-17.66339	-18.19696
3.65	17.95784	14.32540	8.80552	2.12563	-4.83434	-11.15761	-16.01041	-18.75430	-19.02671
3.70	18.84359	14.92543	8.98715	1.83256	-5.57003	-12.21873	-17.21375	-19.87902	-19.85355
3.75	19.77199	15.54752	9.16184	1.50314	-6.36465	-13.34785	-18.47581	-21.03608	-20.67251
3.80	20.74572	16.19250	9.32871	1.13417	-7.22211	-14.54809	-19.79852	-22.22414	-21.47912
3.85	21.76625	16.86075	9.48736	0.72313	-8.14585	-15.82244	-21.18216	-23.41778	-22.26815
3.90	22.83704	17.55315	9.63323	0.26665	-9.14015	-17.17414	-22.62890	-24.68542	-23.03433
3.95	23.95877	18.27058	9.76776	-0.23881	-10.20878	-18.60674	-24.13866	-25.95365	-23.77083
4.00	25.13565	19.01316	9.88877	-0.79686	-11.35670	-20.12331	-25.71311	-27.24325	-24.47251
4.05	26.38305	19.78177	9.99392	-1.41098	-12.58750	-21.72760	-27.35201	-28.55126	-25.13082
4.10	27.66204	20.57702	10.08130	-2.08569	-13.90673	-23.42279	-29.05613	-29.87351	-25.73892
4.15	29.01701	21.39985	10.14882	-2.82496	-15.31867	-25.21255	-30.82539	-31.20590	-26.12927
4.20	30.43847	22.25023	10.19471	-3.63325	-16.82951	-27.10023	-32.66062	-32.54361	-26.77021
4.25	31.92756	23.12937	10.21578	-4.51527	-18.44307	-29.08957	-34.55984	-33.88206	-27.17503
4.30	33.48928	24.03792	10.20996	-5.47673	-20.16644	-31.18438	-36.52442	-35.21501	-27.49380
4.35	35.12521	24.97606	10.17433	-6.52260	-22.00409	-33.38766	-38.55151	-36.53586	-27.71408
4.40	36.84079	25.94430	10.10571	-7.65848	-23.96331	-35.70350	-40.64193	-37.83841	-27.82675
4.45	38.63817	26.94355	10.00072	-8.89029	-26.04930	-38.13518	-42.79222	-39.11504	-27.81811
4.50	40.52296	27.97432	9.85611	-10.22468	-28.26954	-40.68587	-45.00121	-40.35690	-27.67685
4.55	42.49703	29.03708	9.66751	-11.66829	-30.63003	-43.35051	-47.26550	-41.55555	-27.38829
4.60	44.56695	30.13175	9.43215	-13.22809	-33.13885	-46.15954	-49.58444	-42.70082	-26.93962
4.65	46.73528	31.25916	9.14460	-14.91193	-35.80139	-49.08927	-51.95098	-43.78207	-26.31484
4.70	49.00875	32.42019	8.80054	-16.72708	-38.62729	-52.15078	-54.46493	-44.78877	-25.50014
4.75	51.38955	33.61402	8.39541	-18.68222	-41.62283	-55.34860	-56.81814	-45.70843	-24.47719
4.80	53.88572	34.84139	7.92303	-20.78564	-44.79654	-58.68340	-59.30666	-46.52672	-23.23084
4.85	56.49994	36.10361	7.37911	-23.04742	-48.15772	-62.16105	-61.82639	-47.23167	-21.74264
4.90	59.24067	37.39897	6.75641	-25.47672	-51.71414	-65.78156	-64.36824	-47.80653	-19.99464
4.95	62.11104	38.72870	6.04834	-28.08394	-55.47333	-69.54706	-66.92416	-48.23565	-17.96785
5.00	65.11925	40.09199	5.24913	-30.87966	-59.44716	-73.46043	-69.48834	-48.50219	-15.64204
5.05	68.26951	41.48935	4.34973	-33.87499	-63.64207	-77.52282	-72.04822	-48.58808	-12.99692
5.10	71.57158	42.92048	3.34510	-37.08234	-68.07066	-81.73482	-74.59685	-48.47241	-10.01152
5.15	75.02932	44.38410	2.22343	-40.51314	-72.13951	-86.09819	-77.11912	-48.13608	-6.66598
5.20	78.65344	45.88042	0.97884	-44.18195	-77.66282	-90.61251	-79.60781	-47.55714	-2.93471
5.25	82.44813	47.40872	-0.40018	-48.10236	-82.84640	-95.27670	-82.04319	-46.71222	1.20149
5.30	86.42415	48.96811	-1.92318	-52.28761	-88.30426	-100.09179	-84.41582	-45.57649	5.76755
5.35	90.58638	50.55720	-3.60130	-56.75312	-94.04447	-105.05476	-86.70498	-44.12540	10.78600
5.40	94.94931	52.17668	-5.44427	-61.51652	-100.08141	-110.16516	-88.89840	-42.33204	16.28189
5.45	99.51624	53.82317	-7.46674	-66.59204	-106.42205	-115.41915	-90.96486	-40.16633	22.27687
5.50	104.30064	55.49520	-9.67985	-71.99940	-113.08248	-120.82217	-92.90797	-37.59996	28.79731
5.55	109.31016	57.19076	-12.74078	-77.75530	-120.06896	-126.34031	-94.68256	-34.60179	35.86609
5.60	114.55917	58.90888	-14.73790	-83.87971	-127.33884	-131.99873	-96.27454	-31.13917	43.50831
5.65	120.05195	60.36286	-17.60880	-90.39219	-135.07799	-137.35514	-97.66570	-27.17793	52.45681
5.70	125.80683	62.40132	-20.73369	-97.31462	-143.12450	-143.67740	-98.80147	-22.68378	60.60664
5.75	131.82992	64.17122	-24.12803	-104.66842	-151.54265	-149.68205	-99.67912	-17.61806	70.10779
5.80	138.13992	65.95031	-27.81097	-112.47588	-160.35236	-155.78106	-100.25986	-11.94517	80.27692
5.85	144.74345	67.73618	-31.79946	-120.76017	-169.55661	-161.96613	-100.50616	-5.62168	92.02674
5.90	151.66098	69.52451	-36.11836	-129.54900	-179.17729	-168.22260	-100.38820	-1.38896	102.69704
5.95	158.91886	71.31240	-40.78503	-138.86788	-189.21598	-174.53559	-99.86154	9.13135	114.98707
6.00	166.48194	73.09278	-45.82923	-148.74127	-199.69451	-180.88829	-98.89189	17.64990	128.02532
6.05	174.41552	74.85978	-51.27074	-159.20055	-210.61336	-187.26183	-97.43095	26.98616	142.88289
6.10	182.72274	76.60912	-57.13861	-170.27668	-221.99369	-193.63728	-95.43537	37.71910	156.40166
6.15	191.41393	78.33169	-63.45936	-181.99510	-233.83501	-199.99143	-92.85484	48.30579	171.65394
6.20	200.51847	80.02212	-70.26326	-194.39116	-246.15164	-206.29593	-89.64054	60.38368	187.93125
6.25	210.04306	81.66960	-77.57990	-207.49963	-258.96660	-212.63320	-85.73602	73.46949	204.89921
6.30	220.01447	83.27021	-85.44577	-221.35412	-272.27414	-218.65849	-81.08648	87.61488	222.68014
6.35	230.73290	84.91484	-94.01163	-236.28727	-286.45433	-224.92918	-75.72347	103.00238	241.56937
6.40	241.36929	86.28027	-102.96036	-251.44715	-300.40710	-230.46689	-69.30226	119.29138	260.66739
6.45	253.10847	87.77662	-112.82488	-268.09688	-315.64004	-236.36844	-62.11387	137.09078	281.21333
6.50	264.75237	88.96249	-123.11078	-284.97596	-330.61787	-241.42258	-53.76993	155.81572	301.85056
6.55	277.61038	90.26258	-134.44346	-303.51120	-346.94296	-246.78423	-44.47143	176.24394	324.00713
6.60	290.35838	91.21779	-146.23956	-322.27248	-362.94522	-251.17421	-33.90988	197.60057	346.11525

TABLE 2

Function $\Phi_2(z) = \text{ch}\, z \cos \gamma z$, where $z = \frac{1}{\alpha}\eta$

z \ γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	1	1	1	1	1	1	1	1	1
0.05	1.00119	1.00115	1.00109	1.00100	1.00088	1.00075	1.00058	1.00040	1.00017
0.10	1.00495	1.00480	1.00460	1.00419	1.00381	1.00320	1.00253	1.00179	1.00092
0.15	1.01119	1.01084	1.01027	1.00948	1.00844	1.00720	1.00571	1.00402	1.00209
0.20	1.01987	1.01925	1.01823	1.01681	1.01500	1.01273	1.01010	1.00704	1.00359
0.25	1.03108	1.03012	1.02850	1.02625	1.02334	1.01982	1.01564	1.01085	1.00540
0.30	1.04487	1.04346	1.04111	1.03782	1.03360	1.02845	1.02237	1.01538	1.00747
0.35	1.06122	1.05928	1.05503	1.05050	1.04566	1.03857	1.02918	1.02053	1.00966
0.40	1.08020	1.07761	1.07330	1.06726	1.05952	1.05009	1.03897	1.02619	1.01177
0.45	1.10184	1.09850	1.09292	1.08515	1.07515	1.06301	1.04868	1.03227	1.01373
0.50	1.12622	1.12199	1.11497	1.10516	1.09257	1.07727	1.05926	1.03861	1.01537
0.55	1.15334	1.14812	1.14770	1.12726	1.11169	1.09277	1.07052	1.04508	1.01644
0.60	1.18334	1.16509	1.16631	1.15149	1.13253	1.10948	1.08244	1.05150	1.01679
0.65	1.21620	1.20850	1.19568	1.17782	1.15497	1.12727	1.09478	1.05769	1.01610
0.70	1.25209	1.24289	1.22759	1.20629	1.17907	1.1460[illegible]	1.10747	1.06345	1.01421
0.75	1.29103	1.28014	1.26203	1.23686	1.20470	1.16580	1.12031	1.06855	1.01076
0.80	1.33315	1.32035	1.31247	1.26954	1.23185	1.18629	1.13315	1.07275	1.00550
0.85	1.37852	1.36358	1.33877	1.29049	1.26043	1.20746	1.14576	1.07579	0.99805
0.90	1.42730	1.40993	1.38117	1.34123	1.29043	1.22918	1.15798	1.07741	0.98811
0.95	1.47951	1.45948	1.42627	1.38020	1.32167	1.25126	1.16951	1.07728	0.97528
1.00	1.53536	1.51233	1.47417	1.42127	1.35418	1.27356	1.18021	1.07508	0.95919
1.05	1.59494	1.56855	1.52485	1.46440	1.39180	1.29428	1.18972	1.07046	0.93940
1.10	1.65844	1.62831	1.57849	1.50959	1.42245	1.31811	1.19785	1.06310	0.91600
1.15	1.72593	1.69166	1.63501	1.55681	1.45800	1.33998	1.20420	1.05256	0.88698
1.20	1.79764	1.75876	1.69460	1.60692	1.49441	1.36127	1.20854	1.03844	0.85342
1.25	1.87367	1.82971	1.75717	1.65724	1.53143	1.38174	1.20951	1.02031	0.81423
1.30	1.95427	1.90467	1.82291	1.71039	1.56900	1.40114	1.20965	0.99771	0.76895
1.35	2.03958	1.98376	1.89185	1.76545	1.60694	1.41922	1.20564	0.97015	0.71760
1.40	2.12986	2.06714	1.96396	1.82237	1.64509	1.43564	1.19809	0.93710	0.65779
1.45	2.22520	2.15493	2.03938	1.88106	1.68323	1.45012	1.18653	0.89805	0.59072
1.50	2.32599	2.24735	2.11823	1.94154	1.72109	1.46228	1.17049	0.85242	0.51520
1.55	2.43232	2.34450	2.20043	2.00266	1.75278	1.47177	1.14947	0.79961	0.43050
1.60	2.54455	2.44663	2.28618	2.06738	1.79574	1.47822	1.12295	0.73901	0.33615
1.65	2.66282	2.55384	2.37546	2.13259	1.83175	1.48119	1.09039	0.66996	0.23132
1.70	2.78754	2.66640	2.46339	2.19922	1.86663	1.48026	1.05117	0.59180	0.11537
1.75	2.91887	2.78447	2.55015	2.26707	1.90002	1.47480	1.00468	0.50353	−0.01245
1.80	3.05725	2.90828	2.66531	2.33592	1.93163	1.46464	0.95033	0.40528	−0.15283
1.85	3.20288	3.03803	2.76939	2.40633	1.96105	1.44894	0.88733	0.29545	−0.30650
1.90	3.35621	3.17391	2.87739	2.47731	1.98802	1.42721	0.81506	0.17352	−0.47421
1.95	3.51750	3.31625	2.91750	2.54892	2.01203	1.39887	0.73266	0.03872	−0.65672
2.00	3.68722	3.46521	3.10509	2.62116	2.03272	1.36327	0.63946	−0.10986	−0.85477
2.05	3.86738	3.62108	3.20509	2.69362	2.04957	1.31969	0.53452	−0.27303	−1.06913
2.10	4.05326	3.78413	3.34873	2.76616	2.06208	1.26741	0.41704	−0.45169	−1.30053
2.15	4.25047	3.95463	3.47658	2.83855	2.06970	1.20566	0.28606	−0.64673	−1.54971
2.20	4.45782	4.13282	3.60860	2.91044	2.07200	1.13366	0.14065	−0.85913	−1.81748
2.25	4.67565	4.31907	3.74468	2.98160	2.06814	1.05050	−0.04029	−1.08978	−2.10445
2.30	4.90460	4.51360	3.88496	3.05165	2.05765	0.95526	−0.19741	−1.33970	−2.41147
2.35	5.14503	4.71682	4.02922	3.12027	2.03969	0.84700	−0.39224	−1.60989	−2.73904
2.40	5.39769	4.92896	4.17777	3.18702	2.01362	0.72474	−0.60565	−1.90131	−3.08800
2.45	5.66290	5.15044	4.33024	3.25150	1.97850	0.58741	−0.83882	−2.21497	−3.45879
2.50	5.94163	5.38157	4.48693	3.31327	1.93363	0.43380	−1.09308	−2.55195	−3.85212
2.55	6.23417	5.62271	4.64756	3.37186	1.87801	0.26279	−1.36944	−2.91321	−4.26847
2.60	6.54150	5.87428	4.81216	3.42661	1.81071	0.07311	−1.66944	−3.29976	−4.70825
2.65	6.86398	6.13660	4.98063	3.47701	1.73065	−0.13656	−1.99423	−3.71271	−5.17188
2.70	7.20271	6.41007	5.15296	3.52247	1.63676	−0.36755	−2.34525	−4.15301	−5.65980
2.75	7.55808	6.69508	5.32884	3.56225	1.52778	−0.62135	−2.72383	−4.62166	−6.17205
2.80	7.93137	6.99221	5.50837	3.59555	1.32019	−0.89947	−3.13150	−5.11966	−6.70906
2.85	8.32286	7.30164	5.69118	3.62168	1.25998	−1.20335	−3.56964	−5.64799	−7.27050

z \ γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2.90	8.73395	7.62398	5.87735	3.63982	1.09831	−1.53490	−4.03986	−6.20758	−7.85677
2.95	9.16523	7.95970	6.06628	3.64889	0.91626	−1.89562	−4.54358	−6.79918	−8.46720
3.00	9.61804	8.30924	6.25816	3.64812	0.71219	−2.28737	−5.08266	−7.42379	−9.10187
3.05	10.09286	8.67300	6.45230	3.63628	0.48441	−2.71210	−5.65827	−8.08225	−9.75992
3.10	10.59134	9.05157	6.64877	3.61226	0.23122	−3.17174	−6.27241	−8.77497	−10.44120
3.15	11.11410	9.44543	6.84114	3.54787	−0.04910	−3.66827	−6.92636	−9.50295	−11.14426
3.20	11.66298	9.85512	7.04664	3.52283	−0.35877	−4.20388	−7.62215	−10.26660	−11.86866
3.25	12.23835	10.28102	7.24713	3.45464	−0.69971	−4.78071	−8.36114	−11.06636	−12.61339
3.30	12.84227	10.72392	7.44833	3.36898	−1.07390	−5.40113	−9.14532	−11.90262	−13.37535
3.35	13.47556	11.18426	7.64928	3.26401	−1.48425	−6.06743	−9.97597	−12.77553	−14.15447
3.40	14.14006	11.66257	7.84989	3.01361	−1.93244	−6.78213	−10.85519	−13.68545	−14.94879
3.45	14.83603	12.15958	8.04889	2.98988	−2.42151	−7.54769	−11.78403	−14.63202	−15.75535
3.50	15.56801	12.67556	8.24614	2.65115	−2.95411	−8.36662	−12.76488	−15.61524	−16.57216
3.55	16.33465	13.21140	8.44048	2.61716	−3.53263	−9.24202	−13.97270	−16.63464	−17.39593
3.60	17.13893	13.76773	8.63136	2.38835	−4.16066	−10.17641	−14.88737	−17.68978	−18.22415
3.65	17.98212	14.34477	8.81742	2.12851	−4.84088	−11.17270	−16.03206	−18.77966	−19.05244
3.70	18.86664	14.94367	8.99814	1.83480	−5.57684	−12.23368	−17.23481	−19.90333	−19.87783
3.75	19.79175	15.56473	9.17198	1.50480	−6.37169	−13.36262	−18.49626	−20.84664	−20.69539
3.80	20.76649	16.20671	9.33806	1.13531	−7.22934	−14.56266	−19.81351	−22.24639	−21.50063
3.85	21.78597	16.87603	9.49596	0.72379	−8.15323	−15.83678	−21.20135	−23.46251	−22.28832
3.90	22.85577	17.56755	9.64113	0.26688	−9.14765	−17.18823	−22.64746	−24.70567	−23.05322
3.95	23.97654	18.28412	9.77500	−0.23899	−10.21636	−18.62054	−24.15656	−25.97289	−23.78846
4.00	25.15252	19.02592	9.89541	−0.79740	−11.36432	−20.13682	−25.73036	−27.26153	−24.48893
4.05	26.39907	19.79378	9.99995	−1.41183	−12.59510	−21.74072	−27.36852	−28.56850	−25.14599
4.10	27.67724	20.58833	10.08684	−2.08684	−13.91437	−23.43566	−29.07209	−29.83992	−25.75306
4.15	29.03144	21.41049	10.15386	−2.82637	−15.32629	−25.22509	−30.84071	−31.22141	−26.14225
4.20	30.45215	22.26023	10.19930	−3.63489	−16.83708	−27.11242	−32.67531	−32.55825	−26.78225
4.25	31.94968	23.14539	10.22286	−4.51840	−18.45585	−29.10972	−34.58377	−34.25622	−27.19385
4.30	33.50162	24.04677	10.21372	−5.47874	−20.17387	−31.19586	−36.53787	−35.22798	−27.50393
4.35	35.13691	24.98438	10.17771	−6.52477	−22.01142	−33.39873	−38.56435	−36.54802	−27.72331
4.40	36.85189	25.95212	10.10876	−7.66079	−23.97053	−35.71425	−40.65418	−37.84981	−27.83514
4.45	38.64871	26.95090	10.00345	−8.89271	−26.05641	−38.14558	−42.80389	−39.12572	−27.82570
4.50	40.53296	27.98123	9.85854	−10.22721	−28.27652	−40.69591	−45.01232	−40.36686	−27.68368
4.55	42.50653	29.04356	9.66967	−11.67090	−30.63688	−43.36920	−47.27606	−41.56483	−27.39441
4.60	44.07849	30.13784	9.43405	−13.23076	−33.14555	−46.16886	−49.59446	−42.70945	−26.94506
4.65	46.74282	31.31717	9.14627	−14.91466	−35.80794	−49.09825	−51.96047	−43.79007	−26.31965
4.70	49.01685	32.42555	8.80110	−16.72984	−38.63368	−52.15940	−54.37392	−44.79618	−25.50435
4.75	51.39724	33.67685	8.39667	−18.68502	−41.62849	−55.35688	−56.82665	−45.71527	−24.48085
4.80	53.89291	34.84668	7.92423	−20.78880	−44.80331	−58.69229	−59.31568	−46.53374	−23.23436
4.85	56.50687	36.10803	7.38001	−23.01831	−48.16363	−62.16867	−61.83397	−47.23746	−21.74530
4.90	59.24635	37.40256	6.75706	−25.47916	−51.71910	−65.78787	−64.37441	−47.81111	−19.99656
4.95	62.11727	38.73259	6.04894	−28.08676	−55.47889	−69.55404	−66.93088	−48.31108	−17.96965
5.00	65.12517	40.09564	5.24962	−30.88247	−59.45236	−73.42258	−69.49465	−48.50659	−15.64346
5.05	68.27512	41.49276	4.35008	−33.87777	−63.64730	−77.52139	−72.05414	−48.59207	−12.99799
5.10	71.57690	42.92367	3.34535	−37.08509	−68.07572	−81.74089	−74.60239	−48.47601	−10.01227
5.15	75.03436	44.38708	2.22358	−40.51586	−72.74439	−86.10997	−77.12430	−48.13932	−6.66642
5.20	78.65823	45.88321	0.97890	−44.09400	−77.66750	−90.61803	−79.61266	−47.56003	−2.93489
5.25	82.45267	47.41136	0.40020	−48.10501	−82.85097	−95.28195	−82.04771	−46.71480	1.20155
5.30	86.48845	48.97055	1.92328	−52.29021	−88.30866	−100.09677	−84.42003	−45.57876	5.76784
5.35	90.59099	51.61283	−3.60148	−56.75600	−94.04925	−105.06010	−86.70939	−44.12764	10.78655
5.40	94.95319	52.17881	−5.44449	−61.51903	−100.08549	−110.16965	−88.90203	−42.33377	16.28256
5.45	99.51992	53.82515	−7.46702	−66.59450	−106.42598	−115.42341	−90.96822	−40.16782	22.27769
5.50	104.30412	55.49706	−9.68017	−72.00180	−113.08626	−120.81621	−92.91108	−37.60121	27.57979
5.55	109.29647	57.18560	−12.73919	−77.75556	−120.05392	−126.32449	−94.67071	−34.59745	35.86160
5.60	114.56230	58.91049	−14.73709	−83.88201	−127.40232	−132.00234	−96.27717	−31.14003	43.50350
5.65	120.05493	60.36435	−17.60923	−90.39443	−135.08134	−137.35854	−97.65817	−27.17860	52.45811
5.70	125.80964	62.40272	−20.73416	−97.31680	−143.12770	−143.68061	−98.80368	−22.68429	60.60800
5.75	131.83259	64.17252	−24.12851	−104.67054	−151.54511	−149.68508	−99.68114	−17.61842	70.10921
5.80	138.14248	65.95152	−27.81148	−112.47794	−160.35530	−155.78391	−100.26170	−11.94539	80.27839
5.85	144.74585	67.73731	−31.79999	−120.76217	−169.55943	−161.96882	−100.50782	−5.62177	91.13242
5.90	151.66325	69.52556	−36.11890	−129.55094	−179.17998	−168.22513	−100.38970	1.38898	102.69858
5.95	158.92101	71.31337	−40.78558	−139.82915	−189.21854	−174.54715	−99.86289	9.13147	115.98864
6.00	166.48399	73.09368	−45.82979	−148.74309	−199.69647	−180.89052	−98.89312	17.65012	128.02690
6.05	174.41745	74.86061	−51.27131	−159.20231	−210.61570	−187.26390	−97.43203	26.98646	142.88448
6.10	182.72458	76.60990	−57.13919	−170.27840	−221.99593	−193.63923	−95.43634	37.19141	156.40324
6.15	191.41568	78.33241	−63.45994	−181.99676	−233.83714	−199.99324	−92.85568	48.32967	171.76695
6.20	200.52012	80.92277	−70.26384	−194.39276	−246.16366	−206.29763	−89.64128	60.38418	187.93280
6.25	210.04462	81.67021	−77.58048	−207.50117	−258.96859	−212.53110	−85.73662	73.47003	204.90074
6.30	220.01596	83.27077	−85.44634	−221.35561	−272.27598	−218.65997	−81.08703	87.61647	222.68165
6.35	230.74236	84.92843	−94.01549	−236.29696	−286.45174	−224.93840	−75.72658	103.00660	241.57928
6.40	241.38666	86.28648	−102.96777	−251.46525	−300.42874	−230.48348	−69.30725	119.29996	260.68616
6.45	253.11772	87.77983	−112.82901	−268.10668	−315.65198	−236.37709	−62.11614	137.09580	281.22361
6.50	264.75357	88.96289	−123.11134	−284.97725	−330.61937	−241.42368	−53.77017	155.81643	301.85193
6.55	277.61152	90.26295	−134.44401	−303.51245	−346.94439	−246.78524	−44.47161	176.24467	324.00846
6.60	290.35946	91.18137	−146.24010	−322.27367	−362.94656	−251.17514	−33.91000	197.60130	346.11653

TABLE 3

Function $\Phi_3(z) = \operatorname{ch} z \sin \gamma z$, where $z = \bar{\alpha}\gamma$

z \ γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0	0	0	0	0	0	0	0	0
0.05	0.00501	0.01000	0.01502	0.02002	0.02503	0.03004	0.03504	0.04004	0.04503
0.10	0.01001	0.02002	0.03004	0.04003	0.05004	0.06003	0.07002	0.08000	0.09000
0.15	0.01517	0.03034	0.04549	0.06064	0.07577	0.09090	0.10600	0.12106	0.13610
0.20	0.02040	0.04080	0.06116	0.08151	0.10183	0.12211	0.14234	0.16252	0.18290
0.25	0.02578	0.05155	0.07727	0.10139	0.12859	0.15413	0.17958	0.20491	0.22942
0.30	0.03136	0.06268	0.09395	0.12514	0.15621	0.18715	0.21791	0.24848	0.27883
0.35	0.03713	0.07421	0.11118	0.14805	0.18471	0.22116	0.25732	0.29319	0.32869
0.40	0.04323	0.08639	0.12941	0.17224	0.21478	0.25697	0.29876	0.34007	0.38083
0.45	0.04962	0.09913	0.14844	0.19746	0.24607	0.29419	0.34172	0.38854	0.43459
0.50	0.05636	0.11257	0.16851	0.22403	0.27897	0.33323	0.38666	0.43912	0.49048
0.55	0.06350	0.12680	0.18972	0.25208	0.31423	0.37430	0.43381	0.49200	0.54870
0.60	0.07108	0.14191	0.21223	0.28179	0.35033	0.41760	0.48339	0.54695	0.60950
0.65	0.07916	0.15799	0.23615	0.31333	0.38916	0.46337	0.53561	0.60559	0.67300
0.70	0.08779	0.17515	0.26165	0.34688	0.43040	0.51181	0.59072	0.66673	0.73947
0.75	0.09701	0.19348	0.28884	0.38260	0.48067	0.56315	0.64915	0.73103	0.80901
0.80	0.10687	0.21308	0.31791	0.42071	0.52082	0.61760	0.71043	0.79871	0.88187
0.85	0.11744	0.23406	0.34899	0.46139	0.57046	0.67541	0.77547	0.86995	0.95814
0.90	0.12881	0.25657	0.38225	0.50483	0.62335	0.73681	0.83570	0.94495	1.03797
0.95	0.14098	0.28069	0.41787	0.55127	0.67971	0.80201	0.91709	1.02389	1.12145
1.00	0.15405	0.30656	0.45602	0.60091	0.73980	0.87128	0.99408	1.10694	1.20874
1.05	0.16808	0.33433	0.49689	0.65396	0.80382	0.94485	1.07546	1.19424	1.29985
1.10	0.18317	0.36412	0.54067	0.71069	0.87212	1.02300	1.16152	1.28600	1.39493
1.15	0.19938	0.39557	0.58759	0.77132	0.94484	1.10593	1.25236	1.38228	1.49391
1.20	0.21675	0.43039	0.63784	0.83659	1.02237	1.19931	1.34829	1.48327	1.59693
1.25	0.23543	0.46719	0.69167	0.90536	1.10489	1.28722	1.44944	1.58905	1.70384
1.30	0.25549	0.50668	0.74932	0.97930	1.19277	1.38610	1.55603	1.69971	1.81472
1.35	0.27703	0.54902	0.81102	1.05827	1.28623	1.49083	1.66825	1.81536	1.92940
1.40	0.30014	0.59442	0.87075	1.14254	1.38565	1.60165	1.78632	1.93602	2.04785
1.45	0.32496	0.64306	0.94766	1.23241	1.49125	1.71886	1.91032	2.06173	2.16984
1.50	0.35194	0.69518	1.02323	1.32826	1.60350	1.84271	2.04053	2.19254	2.29536
1.55	0.38006	0.75101	1.10335	1.43044	1.72259	1.97348	2.17702	2.32838	2.42390
1.60	0.41064	0.81079	1.19022	1.53926	1.84897	2.11143	2.31998	2.46926	2.55545
1.65	0.44339	0.87475	1.28235	1.65512	1.98290	2.25687	2.46946	2.61507	2.68909
1.70	0.47849	0.94322	1.38073	1.87852	2.12486	2.41004	2.62573	2.76570	2.82597
1.75	0.51609	1.01642	1.48565	1.90959	2.27513	2.57120	2.78871	2.92116	2.96413
1.80	0.55633	1.09467	1.59767	2.04900	2.43417	2.74066	2.95859	3.08093	3.10371
1.85	0.59941	1.17835	1.71705	2.19719	2.60229	2.91866	3.13536	3.24510	3.24403
1.90	0.64547	1.26770	1.84431	2.35454	2.78005	3.10545	3.31913	3.41332	3.38468
1.95	0.69476	1.36316	1.97987	2.52160	2.96770	3.30133	3.50975	3.58526	3.52478
2.00	0.74744	1.46508	2.12420	2.69885	3.16578	3.50652	3.70746	3.76058	3.66382
2.05	0.80376	1.57384	2.27802	2.88681	3.37463	3.72125	3.91192	3.93888	3.80057
2.10	0.86392	1.68988	2.44158	3.08602	3.59486	3.94576	4.12326	4.11961	3.93498
2.15	0.92822	1.81366	2.61562	3.29711	3.82676	4.18030	4.34118	4.30233	4.06527
2.20	0.99685	1.94566	2.80068	3.52067	4.07133	4.25502	4.56576	4.48637	4.18941
2.25	1.07016	2.08636	2.99733	3.75730	4.32771	4.68011	4.79647	4.67114	4.31020
2.30	1.14838	2.23627	3.20639	4.00761	4.59777	4.94579	5.03384	4.85578	4.42248
2.35	1.23183	2.39600	3.42833	4.27237	4.88141	5.22222	5.27587	5.03960	4.52616
2.40	1.32089	2.56609	3.66414	4.55220	5.17930	5.50949	5.52383	5.22139	4.61994
2.45	1.41584	2.74722	3.91439	4.84789	5.49175	5.80767	5.77667	5.38499	4.70219
2.50	1.51713	2.94000	4.18001	5.16013	5.81941	6.11689	6.03410	5.57608	4.77235
2.55	1.62508	3.14514	4.46169	5.48980	6.16272	6.43724	6.29731	5.74634	4.82563
2.60	1.74018	3.36339	4.76051	5.83759	6.52235	6.76860	6.55992	5.91023	4.86326
2.65	1.86272	3.59550	5.07722	6.20452	6.89854	7.11106	6.82699	6.06640	4.88219
2.70	1.99340	3.84241	5.41296	6.59130	7.29201	7.46443	7.09596	6.21329	4.88047
2.75	2.13248	4.10483	5.76855	6.99892	7.70313	7.82870	7.36575	6.34938	4.85568
2.80	2.28072	4.38377	6.14531	7.42828	8.13265	8.20354	7.63550	6.42419	4.80581
2.85	2.43845	4.68011	6.54416	7.88038	8.58070	8.58894	7.90405	6.58162	4.72807

s \ γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2.90	2.60631	4.99497	7.01212	8.35625	9.04813	8.98442	8.17040	6.67944	4.62027
2.95	2.78504	5.32946	7.41330	8.85698	9.53509	9.38967	8.43291	6.74765	4.47931
3.00	2.97519	5.68460	7.88630	9.38346	10.04239	9.80439	8.69050	6.80030	4.30279
3.05	3.17747	6.06174	8.40741	9.93695	10.57013	10.22730	8.94124	6.82963	4.08726
3.10	3.39272	6.46204	8.91522	10.51849	11.11905	10.65962	9.18402	6.83272	3.83013
3.15	3.62280	6.88673	9.47419	11.12942	11.68923	11.09903	9.41633	6.80712	3.52800
3.20	3.86501	7.33759	10.06510	11.77086	12.28137	11.54515	9.63667	6.74979	3.17757
3.25	4.12362	7.81576	10.68938	12.44395	12.89545	11.99711	9.84244	6.65746	2.76630
3.30	4.39876	8.32296	11.34891	13.15014	13.53227	12.45403	10.03188	6.52702	2.31789
3.35	4.69105	8.86085	12.04510	13.89050	14.19129	12.91464	10.20170	6.35494	1.80145
3.40	5.01693	9.43106	12.78058	14.66666	14.87365	13.37783	10.35018	6.13763	1.22240
3.45	5.33208	10.03573	13.55646	15.47992	15.57878	13.84198	10.47372	5.87128	0.57672
3.50	5.68282	10.67654	14.37560	16.33169	16.30749	14.30582	10.56948	5.55173	0.13938
3.55	6.05520	11.35554	15.23956	17.22329	17.05884	14.76745	10.63396	5.17474	-0.92993
3.60	6.45104	12.07508	16.14931	18.15639	17.83390	15.22488	10.66408	4.73605	-1.79923
3.65	6.87121	12.83747	17.11200	19.13232	18.63143	15.67631	10.65544	4.23084	-2.75126
3.70	7.31775	13.64494	18.12539	20.15263	19.45247	16.11940	10.60448	3.65462	-3.79000
3.75	7.79141	14.52132	19.19315	21.21891	20.29548	16.55134	10.50681	3.00195	-4.92028
3.80	8.29413	15.40548	20.31858	22.33293	21.15715	16.96991	10.35775	2.26793	-6.14546
3.85	8.82835	16.36428	21.50389	23.40209	22.04784	17.37203	10.15251	1.44687	-7.47058
3.90	9.39500	17.37900	22.75298	24.70987	22.95586	17.75412	9.88627	0.53352	-8.89930
3.95	9.99607	18.45298	24.06772	25.97627	23.88354	18.11346	9.55316	-0.47824	-10.43586
4.00	10.62437	19.58983	25.45236	27.29649	24.83137	18.44562	9.14798	-1.59398	-12 08444
4.05	11.31131	20.79251	26.90921	28.67270	25.79651	18.74711	8.66419	-2.82051	-13.84876
4.10	12.02942	22.06496	28.44287	30.10630	26.77913	19.01332	8.09597	-4.16372	-15.73292
4.15	12.79100	23.41074	30.05552	31.59861	27.77640	19.23955	7.43599	-5.62999	-17.73992
4.20	13.59906	24.83423	31.75283	33.15189	28.78862	19.42109	6.67780	-7.22642	-19.87432
4.25	14.45965	26.34727	33.54641	34.77736	29.82023	19.55775	5.81528	-8.96175	-22.14415
4.30	15.36444	27.93147	35.41337	36.44720	30.84535	19.62729	4.83596	-10.83586	-24.53475
4.35	16.32743	29.61448	37.38454	38.19239	31.88537	19.64163	3.73663	-12.86318	-27.06657
4.40	17.34920	31.39345	39.45749	40.00451	32.93147	19.58455	2.50703	-15.04787	-29.73649
4.45	18.43200	33.27360	41.63407	41.88585	33.97798	19.45324	1.13856	-17.39791	-32.54482
4.50	19.57979	35.26091	43.92118	43.83700	35.02414	19.23813	-0.37857	-19.91965	-35.49498
4.55	20.79590	37.36032	46.32206	45.85973	36.06181	18.93191	-2.05375	-22.62109	-38.58500
4.60	22.08526	39.57886	48.84427	47.95529	37.09697	18.52585	-3.89670	-25.50886	-41.81798
4.65	23.45114	41.92254	51.49085	50.12536	38.11425	18.10657	-5.91796	-28.59039	-45.19112
4.70	24.89904	44.39814	54.26891	52.37107	39.11529	17.37749	-8.12907	-31.87302	-48.70458
4.75	26.43264	47.01280	57.18267	54.69280	40.09226	16.61533	-10.54092	-35.36335	-52.35493
4.80	28.05744	49.77343	60.24044	57.09250	41.04049	15.71358	-13.16533	-39.06824	-56.14161
4.85	29.77811	52.68911	63.44551	59.56963	41.95315	14.66102	-16.01324	-42.99430	-60.05827
4.90	31.60168	55.76609	66.80650	62.12564	42.82406	13.41497	-19.10081	-47.14769	-64.06083
4.95	33.53285	59.01621	70.33054	64.76305	43.64643	12.05342	-22.43383	-51.53568	-68.26434
5.00	35.57848	62.44545	74.02368	67.47911	44.41243	10.47251	-26.03137	-56.16209	-72.54245
5.05	37.74339	66.06497	77.89192	70.27463	45.11185	8.68769	-29.90528	-61.03382	-76.92302
5.10	40.03759	69.88495	81.94593	73.15075	45.73757	6.68414	-34.07026	-66.15413	-81.40054
5.15	42.46527	73.91524	85.18846	76.10606	46.27788	4.44630	-38.53973	-71.52785	-85.95998
5.20	45.03665	78.16697	90.63344	79.13952	46.72434	1.95689	-43.32810	-77.15816	-90.59175
5.25	47.75722	82.65277	95.28385	82.25162	47.06354	-0.80135	-48.49664	-83.04916	-95.27623
5.30	50.63939	87.38509	100.15287	85.43977	47.28367	-3.84656	-53.92099	-89.20018	-100.00462
5.35	53.68948	92.37488	105.24334	88.70284	47.37108	-7.19770	-59.75513	-95.61516	-104.75155
5.40	56.91811	97.63780	110.57152	92.03831	47.31330	-10.87681	-65.97160	-102.29185	-109.50210
5.45	60.33435	103.18709	116.14032	95.44425	47.09250	-14.90262	-72.57999	-109.22960	-114.22818
5.50	63.95008	109.03777	121.96506	98.91837	46.69534	-19.30040	-79.60328	-116.42637	-118.91003
5.55	67.76486	115.18706	128.02845	102.43947	46.09436	-24.21423	-87.03568	-123.85921	-123.49784
5.60	71.82488	161.70706	134.40917	106.05187	45.29569	-29.29840	-94.94125	-131.58047	-128.02296
5.65	76.10719	128.55961	141.05011	109.70232	44.53431	-34.94838	-103.28721	-139.52487	-132.39191
5.70	80.63981	135.78146	147.99033	113.40351	42.95968	-41.06783	-112.11090	-147.70341	-136.59289
5.75	85.43245	143.39178	155.23061	117.14874	41.38561	-47.68520	-121.42021	-156.10564	-140.66144
5.80	90.50621	151.41070	162.79293	120.92873	39.51245	-54.82858	-131.20694	-164.65584	-144.32736
5.85	95.87047	159.85935	170.67927	124.73804	37.31064	-62.52528	-141.56516	-173.52835	-147.77376
5.90	101.54686	168.75991	178.90985	128.56898	34.75548	-70.80867	-152.43166	-182.51462	-150.88572
5.95	107.54760	178.13375	187.49164	132.40923	31.81720	-79.70995	-163.84076	-191.66148	-153.60215
6.00	113.86041	188.00704	196.44078	136.25085	28.46611	-89.26320	-175.81132	-200.94105	-155.87778
6.05	120.60993	198.40542	205.76382	140.08107	24.66655	-99.50171	-188.34540	-210.33366	-157.64792
6.10	127.70991	209.35357	215.70485	143.88794	20.38918	-110.46182	-201.46854	-219.80676	-158.85069
6.15	135.09903	220.88173	225.60174	147.65368	15.59430	-122.17877	-215.17741	-229.32805	-159.43494
6.20	143.15404	233.01706	236.14356	151.36574	10.24429	-134.69104	-229.48896	-238.86109	-159.31874
6.25	151.54263	245.79282	247.11376	155.00815	4.29693	-148.03825	-244.40454	-248.36736	-158.42968
6.30	160.41509	259.24161	258.53366	158.56081	-2.28993	-162.26119	-259.93321	-257.80393	-156.69565
6.35	170.01148	273.74609	270.75383	162.21269	-9.57294	-177.61825	-276.42594	-264.60020	-154.22759
6.40	179.72337	288.30975	282.78142	165.32624	-17.56606	-193.50657	-292.85399	-276.28706	-150.36333
6.45	190.43458	304.35324	296.07320	168.68285	-26.38906	-210.85911	-310.60286	-285.55051	-145.75911
6.50	201.26899	320.45266	308.94568	171.44064	-35.98424	-228.73275	-328.19493	-292.81035	-139.60024
6.55	213.24250	333.22089	323.20682	174.42085	-46.56147	-248.27300	-347.21743	-302.45527	-132.49976
6.60	225.35119	356.05136	337.20348	176.72456	-57.98074	-268.33229	-365.98986	-309.91302	-123.67632

TABLE 4

Function $\Phi_4(z) = \operatorname{sh} z \sin \gamma z$, where $z = \bar{\alpha}\eta$

z \ γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0	0	0	0	0	0	0	0	0
0.05	0.00025	0.00050	0.00075	0.00100	0.00125	0 00150	0.00175	0.00200	0.00225
0.10	0.00100	0.00200	0.00301	0.00401	0.00501	0 00601	0.00701	0.00801	0.00901
0.15	0.00226	0.00452	0.00677	0.00903	0.01128	0 01354	0.01578	0.01808	0.02027
0.20	0.00403	0.00805	0.01207	0.01609	0.02010	0 02410	0.02809	0.03208	0.03601
0.25	0.00631	0.01262	0.01892	0.02522	0.03149	0 03775	0.04398	0.05019	0.05636
0.30	0.00913	0.01826	0.02737	0.03645	0.04551	0 05452	0.06348	0.07238	0.08122
0.35	0.01250	0.02498	0.03743	0.04984	0.06219	0 07446	0.08664	0.09871	0.11066
0.40	0.01643	0.03282	0.04917	0.06544	0.08160	0 09763	0.11351	0.12921	0.14469
0.45	0.02093	0.04182	0.06262	0.08331	0.10382	0 12412	0.14417	0.16392	0.18335
0.50	0.02604	0.05202	0.07787	0.10353	0.12892	0 15399	0.17868	0.20293	0.22666
0.55	0.03178	0.06347	0.09496	0.12617	0.15728	0 18734	0.21713	0.24626	0.27464
0.60	0.03817	0.07621	0.11398	0.15133	0.18814	0 22427	0.25960	0.29399	0.32733
0.65	0.04525	0.09032	0.13500	0.17912	0.22247	0 26559	0.30619	0.34620	0.38474
0.70	0.05305	0.10586	0.15813	0.20964	0.26012	0 30932	0.35701	0.40295	0.44691
0.75	0.06162	0.12289	0.18346	0.24301	0.30530	0 35768	0.41231	0.46431	0.51384
0.80	0.07097	0.14149	0.21110	0.27937	0.34585	0 41011	0.47175	0.53038	0.58560
0.85	0.08116	0.16176	0.24117	0.31886	0.39423	0 46676	0.53590	0.60120	0.66214
0.90	0.09226	0.18378	0.27380	0.36161	0.44650	0 52777	0.60476	0.67687	0.74350
0.95	0.10430	0.20765	0.30913	0.40782	0.50284	0 59331	0.67844	0.75745	0.82962
1.00	0.11732	0.23348	0.34729	0.45765	0.56343	0 66356	0.75709	0.84304	0.92057
1.05	0.13140	0.26138	0.38847	0.51127	0.62843	0 73870	0.84081	0.93367	1.01624
1.10	0.14663	0.29148	0.43280	0.56891	0.69813	0 81891	0.92980	1.02944	1.11664
1.15	0.16305	0.32401	0.48051	0.63075	0.77265	0 90433	1.02413	1.13037	1.25307
1.20	0.18070	0.35880	0.53174	0.69704	0.85230	0 99531	1.12400	1.23653	1.33128
1.25	0.19971	0.39631	0.58673	0.76801	0.93727	1 09193	1.22954	1.34797	1.44553
1.30	0.22016	0.43662	0.64571	0.84389	1.02784	1 19444	1.34087	1.46468	1.56378
1.35	0.24214	0.47987	0.70888	0.92498	1.12423	1 30306	1.45814	1.58672	1.68639
1.40	0.26573	0.52627	0.77650	1.01154	1.22679	1 41802	1.58152	1.71406	1.81306
1.45	0.29106	0.57598	0.84881	1.10386	1.33570	1 53957	1.71106	1.84668	1.94351
1.50	0.31820	0.62904	0.92617	1.20228	1.45140	1 66793	1.84698	1.98457	2.07764
1.55	0.34729	0.68627	1.00877	1.30711	1.57407	1 80333	1.98933	2.12764	2.23753
1.60	0.37879	0.74791	1.09791	1.41988	1.70557	1 94768	2.14005	2.27776	2.35726
1.65	0.41185	0.81252	1.19112	1.53737	1.84183	2.09631	2.29377	2.42903	2.49778
1.70	0.44759	0.88229	1.29154	1.66355	1.98761	2.25437	2.45612	2.58706	2.64343
1.75	0.48584	0.95683	1.39855	1.79764	2.14175	2.42046	2.62522	2.74981	2.79035
1.80	0.52674	1.03635	1.51269	1.94001	2.30469	2.59488	2.80121	2.91704	2.93861
1.85	0.57048	1.12149	1.63419	2.09117	2.47672	2.77782	2.98406	3.08851	3.08749
1.90	0.61722	1.21222	1.76360	2.25150	2.65839	2.96955	3.17387	3.26394	3.23656
1.95	0.66719	1.30907	1.90131	2.42154	2.84994	3.17033	3.37049	3.44300	3.38492
2.00	0.72055	1.41237	2.04787	2.60176	3.05189	3.38038	3.57409	3.62530	3.53202
2.05	0.77755	1.52252	2.20374	2.79268	3.26459	3.59992	3.78436	3.81045	3.67665
2.10	0.83840	1.63995	2.36944	2.99484	3.48864	3.82917	4.00143	3.99789	3.81872
2.15	0.90336	1.76511	2.54559	3.20884	3.72431	4.06838	4.22496	4.18715	3.95643
2.20	0.97267	1.89846	2.73274	3.43527	3.97258	4.31769	4.45501	4.37755	4.08779
2.25	1.04665	2.04052	2.93146	3.67473	4.23261	4.57727	4.69108	4.56849	4.21548
2.30	1.12552	2.19176	3.14257	3.92784	4.50626	4.84735	4.93316	4.75913	4.33445
2.35	1.20963	2.35281	3.36653	4.19535	4.79341	5.12808	5.18076	4.94875	4.44456
2.40	1.29982	2.46953	3.60432	4.47788	5.09474	5.41955	5.43365	5.13634	4.54451
2.45	1.39491	2.70661	3.85652	4.77623	5.41057	5.72181	5.69128	5.30538	4.63268
2.50	1.49683	2.90065	4.12406	5.09106	5.74152	6.03501	5.95334	5.50145	4.70748
2.55	1.60538	3.10703	4.40761	5.42326	6.08803	6.35923	6.22099	5.67670	4.76714
2.60	1.72108	3.32648	4.70827	5.77353	6.45077	6.69433	6.48793	5.84537	4.80989
2.65	1.84422	3.55978	5.02678	6.14289	6.83001	7.04041	6.75917	6.00613	4.88369
2.70	1.97547	3.80785	5.36428	6.53202	7.22644	7.39730	7.03217	6.15742	4.83650
2.75	2.11512	4.07141	5.72159	6.94195	7.64042	7.76497	7.30579	6.29769	4.81615
2.80	2.26392	4.35147	6.10003	7.37355	8.07273	8.14310	7.57925	6.42509	4.77040
2.85	2.42217	4.64886	6.50047	7.82777	8.52342	8.53160	7.85128	6.53768	4.69650

z \ γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2.90	2.59058	4.96482	6.92449	8.30580	8.99352	8.93019	8.13014	6.63368	4.59238
2.95	2.76982	5.30034	7.37280	8.80858	9.48297	9.33837	8.38683	6.71078	4.45483
3.00	2.96048	5.65649	7.84730	9.33705	9.99272	9.75590	8.64752	6.76666	4.28144
3.05	3.16325	6.03461	8.36978	9.89247	10.52282	10.18152	8.90122	6.79906	4.06896
3.10	3.37898	6.43586	8.87910	10.47588	11.07401	10.61644	9.14682	6.80504	3.81462
3.15	3.60835	6.86148	9.43946	11.08862	11.64638	11.05834	9.38181	6.78217	3.51506
3.20	3.85219	7.31324	10.03170	11.73180	12.24061	11.50684	9.60469	6.72740	3.16703
3.25	4.11124	7.79229	10.65728	12.40659	12.85672	11.96108	9.81289	6.63746	2.76700
3.30	4.38681	8.30034	11.31807	13.11441	13.49551	12.42019	10.00463	6.50928	2.31159
3.35	4.67951	8.83906	12.01548	13.85634	14.15639	12.88289	10.17661	6.33932	1.79702
3.40	5.00576	9.41007	12.75213	14.63403	14.84065	13.34805	10.32715	6.12397	1.21968
3.45	5.32135	10.01552	13.52915	15.44874	15.54740	13.81410	10.45262	5.85946	0.57556
3.50	5.67247	10.65709	14.34941	16.30193	16.27778	14.27976	10.55023	5.54161	- 0.13912
3.55	6.04522	11.33682	15.21444	17.19490	17.03071	14.74310	10.61643	5.16621	- 0.92840
3.60	6.44142	12.05707	16.12704	18.12930	17.80729	15.20216	10.64817	4.72899	- 1.79655
3.65	6.86193	12.82013	17.08890	19.10649	18.60628	15.65515	10.64106	4.22512	- 2.74754
3.70	7.30881	13.62827	18.10325	20.12802	19.42871	16.09971	10.59152	3.65016	- 3.78537
3.75	7.78279	14.48402	19.17192	21.19544	20.27304	16.53304	10.49520	2.99863	- 4.91484
3.80	8.28613	15.39007	20.29826	22.31059	21.13598	16.95293	10.34739	2.26566	- 6.13962
3.85	8.82036	16.34947	21.48442	23.47485	22.02788	17.35631	10.14332	1.44556	- 7.46382
3.90	9.38731	17.36476	22.73933	24.68969	22.93704	17.73957	9.87817	0.53308	- 8.89201
3.95	9.98866	18.43931	24.04988	25.95702	23.86584	18.10004	9.54608	-0.47789	-10.42813
4.00	10.62724	19.57670	25.43530	27.27818	24.81472	18.43325	9.14185	-1.59291	-12.07633
4.05	11.30444	20.77989	26.89297	28.65540	25.78094	18.73579	8.65896	- 2.81880	-13.84040
4.10	12.02282	22.05284	28.42725	30.08977	26.76443	19.00288	8.09152	- 4.16143	-15.72428
4.15	12.78480	23.39911	30.04059	31.68292	27.76260	19.22999	7.43230	- 5.62710	-17.73110
4.20	13.59295	24.82307	31.73856	33.13699	28.77568	19.41236	6.67480	- 7.22317	-19.86539
4.25	14.44964	26.32903	33.52319	34.75329	29.79960	19.54421	5.81126	- 8.95554	-22.12882
4.30	15.35879	27.92118	35.40033	36.43378	30.83400	19.62006	4.83418	-10.83187	-24.52572
4.35	16.32199	29.60462	37.37210	38.17967	31.87475	19.63404	3.73539	-12.85889	-27.05756
4.40	17.34398	31.38399	39.44560	39.99246	32.92155	19.57865	2.50627	-15.04334	-29.72753
4.45	18.42697	33.26452	41.62271	41.87442	33.96871	19.44793	1.13825	-17.39317	-32.53594
4.50	19.57496	35.25221	43.91034	43.82618	35.01549	19.23339	- 0.37847	-19.91473	-35.48622
4.55	20.79078	37.35197	46.31171	45.84948	36.05376	18.92768	- 2.05329	-22.61694	-38.57638
4.60	22.08080	39.57086	48.83440	47.94560	37.08947	18.52211	- 3.89591	-25.50371	-41.80953
4.65	23.44685	41.91487	51.48143	50.11620	38.10728	18.00736	- 5.91688	-28.58517	-45.18286
4.70	24.89493	44.39080	54.25994	52.36241	39.10882	17.37461	- 8.12772	-31.86775	-48.69652
4.75	26.42868	47.00577	57.17411	54.68461	40.08625	16.61284	-10.53934	-35.35806	-52.34710
4.80	28.05318	49.76587	60.23129	57.08332	41.03426	15.71119	-13.16333	-39.06232	-56.13309
4.85	29.77446	52.68266	63.43774	59.56233	41.94801	14.65923	-16.01128	-42.98903	-60.05091
4.90	31.59865	55.76074	66.80009	62.11968	42.81995	13.44368	-19.09696	-47.14317	-64.09497
4.95	33.53372	59.01029	70.32348	64.75653	43.64205	12.05221	-22.43158	-51.53051	-68.25749
5.00	35.57524	62.43977	74.01696	67.47298	44.40839	10.47156	-26.02900	-56.15699	-72.61007
5.05	37.74028	66.05954	77.88556	70.26885	45.10814	8.68697	-29.90282	-61.02880	-76.91670
5.10	40.03462	69.87976	81.93984	73.14581	45.73417	6.68364	-34.06773	-66.14922	-81.39449
5.15	42.46241	73.91028	86.18268	76.10094	46.27475	4.44600	-38.53714	-71.52304	-85.95421
5.20	45.03390	78.16221	90.62792	79.13470	46.72150	1.95677	-43.32546	-77.15346	-90.58623
5.25	47.75459	82.64822	95.27860	82.24709	47.06095	- 0.80131	-48.49397	-83.04459	-95.27098
5.30	50.63687	87.38073	100.14788	85.43551	47.28131	- 3.84637	-53.91831	-89.19574	-99.99963
5.35	53.68675	92.37019	105.23799	88.69833	47.36868	- 7.19733	- 59.75210	- 95.61030	-104.74623
5.40	56.91579	97.63381	110.59670	92.03456	47.31137	- 10.87637	- 65.96891	-102.28768	-109.49763
5.45	60.33213	103.18328	116.13603	95.44072	47.09076	- 14.90206	- 72.57731	-109.22556	-114.22396
5.50	63.94794	109.03412	121.96098	98.91506	46.69378	- 19.29975	- 80.82406	-116.42247	-118.90605
5.55	67.51612	115.20209	128.04446	102.45231	46.10013	- 24.08865	- 87.04658	-123.87473	-123.51331
5.60	71.82292	121.70374	134.40549	106.04897	45.29445	- 29.29760	- 94.93865	-131.57687	-128.01946
5.65	76.10530	128.55643	141.04662	109.69961	44.33370	- 34.94752	-103.28465	-139.52141	-132.38863
5.70	80.63800	135.77843	147.98702	113.40098	42.93872	- 41.06691	-112.10839	-147.70011	-136.60478
5.75	85.27362	143.38889	155.22747	117.14637	41.38177	- 47.68423	-121.41775	-156.10248	-140.65859
5.80	90.50455	151.40793	162.78995	120.92652	39.51172	- 54.82757	-131.19874	-164.71558	-144.32472
5.85	95.86888	159.85670	170.67644	124.73597	37.31002	- 62.52424	-141.56281	-173.52547	-147.77131
5.90	101.54536	168.75738	178.90717	128.56705	34.75496	- 70.82586	-152.42937	-182.51188	-150.88345
5.95	107.54614	178.13134	187.48910	132.40744	31.81677	- 79.70877	-163.83854	-191.65855	-153.60007
6.00	113.85901	188.00473	196.43836	136.24917	28.46576	- 89.26211	-175.80916	-200.93858	-155.87586
6.05	120.60860	198.40322	205.76154	140.07952	24.60628	- 99.50061	-188.34331	-210.33133	-157.64618
6.10	127.70862	209.35145	215.47974	143.88649	20.38897	-110.46070	-201.46650	-219.80454	-158.85600
6.15	135.09780	220.87972	225.59969	147.65234	15.59415	-122.17766	-215.17546	-229.32597	-159.43350
6.20	143.15286	233.01514	236.14162	151.36449	10.24421	-134.68993	-229.48707	-238.85912	-159.34743
6.25	151.54150	245.79099	247.11192	155.00699	4.27100	-148.03721	-244.40282	-248.36551	-158.42845
6.30	160.41400	259.23985	258.53191	158.55974	- 7.73562	-162.26010	-258.84232	-257.80219	-156.69459
6.35	170.00452	273.73487	270.74273	162.20604	- 9.57255	-177.61097	-276.41461	-267.45539	-154.22127
6.40	179.71043	288.28900	282.76107	165.31433	-17.56480	-193.49264	-292.83290	-276.26717	-150.35251
6.45	190.42762	304.34211	296.06237	168.67668	-26.38810	-210.85140	-310.59151	-285.54008	-145.75378
6.50	201.26807	320.45121	308.94428	171.43987	-35.98408	-228.73171	-328.19344	-293.80902	-139.59961
6.55	213.24162	338.21950	323.20549	174.42014	-46.56128	-248.27198	-347.21600	-302.45053	-132.49713
6.60	225.35036	356.05004	337.20224	176.72390	-57.98052	-268.33130	-365.98851	-309.91188	-123.67586

TABLE 5 $X(\xi) = \sin \mu\xi$. Both beam ends simply supported (case 1)

$\mu_1 = 3.1416$ First term of expansion			$\mu_2 = 6.2832$ Second term of expansion		
$\xi = \frac{x}{b}$	$X = -\left(\frac{b}{\mu_1}\right)^2 X''$	$\frac{b}{\mu_1}X' = -\left(\frac{b}{\mu_1}\right)^3 X'''$	$\xi = \frac{x}{b}$	$X = -\left(\frac{b}{\mu_2}\right)^2 X''$	$\frac{b}{\mu_2}X' = -\left(\frac{b}{\mu_2}\right)^3 X'''$
0.0	0.0000	1.0000	0.0	0.0000	1.0000
0.1	0.3090	0.9511	0.1	0.5878	0.8090
0.2	0.5878	0.8090	0.2	0.9511	0.3090
0.3	0.8090	0.5878	0.3	0.9511	−0.3090
0.4	0.9511	0.3090	0.4	0.5878	−0.8090
0.5	1.0000	0.0000	0.5	0.0000	−1.0000
0.6	0.9511	−0.3090	0.6	−0.5878	−0.8090
0.7	0.8090	−0.5878	0.7	−0.9511	−0.3090
0.8	0.5878	−0.8090	0.8	−0.9511	0.3090
0.9	0.3090	−0.9511	0.9	−0.5878	0.8090
1.0	0.0000	−1.0000	1.0	0.0000	1.0000

$\mu_3 = 9.4248$ Third term of expansion			$\mu_4 = 12.5664$ Fourth term of expansion		
$\xi = \frac{x}{b}$	$X = -\left(\frac{b}{\mu_3}\right)^2 X''$	$\frac{b}{\mu_3}X' = -\left(\frac{b}{\mu_3}\right)^3 X'''$	$\xi = \frac{x}{b}$	$X = -\left(\frac{b}{\mu_4}\right)^2 X''$	$\frac{b}{\mu_4}X' = -\left(\frac{b}{\mu_4}\right)^3 X'''$
0.0	0.0000	1.0000	0.0	0.0000	1.0000
0.1	0.8090	0.5878	0.1	0.9511	0.3090
0.2	0.9511	−0.3090	0.2	0.5878	−0.8090
0.3	0.3090	−0.9511	0.3	−0.5878	−0.8090
0.4	−0.5878	−0.8090	0.4	−0.9511	0.3090
0.5	−1.0000	0.0000	0.5	0.0000	1.0000
0.6	−0.5878	0.8090	0.6	0.9511	0.3090
0.7	0.3090	0.9511	0.7	0.5878	−0.8090
0.8	0.9511	0.3090	0.8	−0.5878	−0.8090
0.9	0.8090	−0.5878	0.9	−0.9511	0.3090
1.0	0.0000	−1.0000	1.0	0.0000	1.0000

TABLE 6

$X(\xi) = \sin \mu\xi - \mathrm{ch}\, \mu\xi - \alpha (\cos \mu\xi - \mathrm{ch}\, \mu\xi)$. Both beam ends built in (case 2)

$\mu_1 = 4.7300$ First term of expansion ($\alpha_1 = 1.01781$)					$\mu_2 = 7.8532$ Second term of expansion ($\alpha_2 = 0.999223$)				
$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_1} X'$	$\left(\frac{b}{\mu_1}\right)^2 X''$	$\left(\frac{b}{\mu_1}\right)^3 X'''$	$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_2} X'$	$\left(\frac{b}{\mu_2}\right)^2 X''$	$\left(\frac{b}{\mu_2}\right)^3 X'''$
0.0	0.0000	0.0000	2.0356	−2.0000	0.0	0.0000	0.0000	1.9984	−2.0000
0.1	0.1925	0.7395	1.0934	−1.9682	0.1	0.4551	0.9569	0.4551	−1.8705
0.2	0.6304	1.0416	0.1989	−1.7792	0.2	1.2053	0.7902	−0.7931	−1.2099
0.3	1.1155	0.9494	−0.5536	−1.3636	0.3	1.5044	−0.0978	−1.3229	−0.1001
0.4	1.4814	0.5570	−1.0591	−0.7431	0.4	1.0351	−1.0506	−0.9665	0.9462
0.5	1.6164	0.0000	−1.2372	0.0000	0.5	0.0000	−1.4530	0.0000	1.3742
0.6	1.4814	−0.5570	−1.0591	0.7431	0.6	−1.0351	−1.0506	0.9665	0.9462
0.7	1.1155	−0.9494	−0.5536	1.3636	0.7	−1.5044	−0.0978	1.3229	−0.1001
0.8	0.6304	−1.0416	0.1989	1.7792	0.8	−1.2053	0.7902	0.7931	−1.2099
0.9	0.1925	−0.7395	1.0934	1.9682	0.9	−0.4551	0.9569	−0.4551	−1.8705
1.0	0.0000	0.0000	2.0356	2.0000	1.0	0.0000	0.0000	−1.9984	−2.0000

$\mu_3 = 10.9960$ Third term of expansion ($\alpha_3 = 1.000034$)					$\mu_4 = 14.1370$ Fourth term of expansion ($\alpha_4 = 0.99999$)				
$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_3} X'$	$\left(\frac{b}{\mu_3}\right)^2 X''$	$\left(\frac{b}{\mu_3}\right)^3 X'''$	$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_4} X'$	$\left(\frac{b}{\mu_4}\right)^2 X''$	$\left(\frac{b}{\mu_4}\right)^3 X'''$
0.0	0.0000	0.0000	2.0000	−2.0000	0.0	0.0000	0.0000	2.0000	−2.0000
0.1	0.7705	1.0120	−0.1047	−1.6777	0.1	1.0711	0.9031	−0.5829	−1.3913
0.2	1.5080	0.1094	−1.2861	−1.3307	0.2	1.3244	−0.6923	−1.2052	0.5730
0.3	0.8671	−1.1817	−0.7924	1.1088	0.3	−0.4075	−1.3644	0.4365	1.3352
0.4	−0.6306	−1.2699	−0.6579	1.2481	0.4	−1.3966	0.1967	1.4033	−0.2042
0.5	−1.4060	0.0000	1.4224	0.0000	0.5	0.0000	1.4123	0.0000	−1.4156
0.6	−0.6306	1.2699	0.6579	−1.2481	0.6	1.3966	0.1967	−1.4033	−0.2042
0.7	0.8671	1.1817	−0.7924	−1.1088	0.7	0.4075	−1.3644	−0.4365	1.3352
0.8	1.5080	−0.1094	−1.2861	1.3307	0.8	−1.3244	−0.6923	1.2052	0.5730
0.9	0.7705	−1.0120	−0.1047	1.5777	0.9	−1.0711	0.9031	0.5829	−1.3913
1.0	0.0000	0.0000	2.0000	2.0000	1.0	0.0000	0.0000	−2.0000	−2.0000

TABLE 7

$X(\xi) = \sin \mu\xi + \operatorname{ch} \mu\xi - \alpha(\cos \mu\xi + \operatorname{ch} \mu\xi)$. Both beam ends free (case 3)

$\mu_1 = 4.7300$
First term of expansion ($\alpha_1 = 1.01781$)

$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_1} X'$	$\left(\frac{b}{\mu_1}\right)^2 X''$	$\left(\frac{b}{\mu_1}\right)^3 X'''$
0.0	−2.0356	2.0000	0.0000	0.0000
0.1	−1.0934	1.9682	−0.1925	−0.7395
0.2	−0.1989	1.7792	−0.6304	−1.0416
0.3	0.5536	1.3636	−1.1155	−0.9494
0.4	1.0591	0.7431	−1.4814	−0.5570
0.5	1.2372	0.0000	−1.6164	0.0000
0.6	1.0591	−0.7431	−1.4814	0.5570
0.7	0.5536	−1.3636	−1.1155	0.9494
0.8	−0.1989	−1.7792	−0.6304	1.0416
0.9	−1.0934	−1.9682	−0.1925	0.7395
1.0	−2.0356	−2.0000	0.0000	0.0000

$\mu_2 = 7.8532$
Second term of expansion ($\alpha_2 = 0.999223$)

$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_2} X'$	$\left(\frac{b}{\mu_2}\right)^2 X''$	$\left(\frac{b}{\mu_2}\right)^3 X'''$
0.0	−1.9984	2.0000	0.0000	0.0000
0.1	−0.4551	1.8705	−0.4551	−0.9569
0.2	0.7931	1.2099	−1.2053	−0.7902
0.3	1.3229	0.1001	−1.5044	0.0978
0.4	0.9665	−0.9462	−1.0351	1.0506
0.5	0.0000	−1.3742	0.0000	1.4530
0.6	−0.9665	−0.9462	1.0351	1.0506
0.7	−1.3229	0.1001	1.5044	0.0978
0.8	−0.7931	1.2009	1.2053	−0.7902
0.9	0.4551	1.8705	0.4551	−0.9569
1.0	1.9984	2.0000	0.0000	0.0000

$\mu_3 = 10.996$
Third term of expansion ($\alpha_3 = 1.000034$)

$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_3} X'$	$\left(\frac{b}{\mu_3}\right)^2 X''$	$\left(\frac{b}{\mu_3}\right)^3 X'''$
0.0	−2.0000	2.0000	0.0000	0.0000
0.1	0.1047	1.6777	−0.7705	−1.0120
0.2	1.2861	0.3307	−1.5080	−0.1094
0.3	0.7924	−1.1088	−0.8671	1.1817
0.4	−0.6579	−1.2481	0.6306	1.2699
0.5	−1.4224	0.0000	1.4060	0.0000
0.6	−0.6579	1.2481	0.6306	−1.2699
0.7	0.7924	1.1088	−0.8671	−1.1817
0.8	1.2861	−0.3307	−1.5080	0.1094
0.9	0.1047	−1.6777	−0.7705	1.0120
1.0	−2.0000	−2.0000	0.0000	0.0000

$\mu_4 = 14.137$
Fourth term of expansion ($\alpha_4 = 0.999999$)

$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_4} X'$	$\left(\frac{b}{\mu_4}\right)^2 X''$	$\left(\frac{b}{\mu_4}\right)^3 X'''$
0.0	−2.0000	2.0000	0.0000	0.0000
0.1	0.5829	1.3913	−1.0711	−0.9031
0.2	1.2052	−0.5730	−1.3244	0.6923
0.3	−0.4365	−1.3352	0.4075	1.3644
0.4	−1.4033	0.2042	1.3966	−0.1967
0.5	0.0000	1.4156	0.0000	−1.4123
0.6	1.4033	0.2042	−1.3966	−0.1967
0.7	0.4365	−1.3352	−0.4075	1.3644
0.8	−1.2052	−0.5730	1.3244	0.6923
0.9	−0.5829	1.3913	1.0711	−0.9031
1.0	2.0000	2.0000	0.0000	0.0000

TABLE 8

$X(\xi) = \sin \mu\xi - \operatorname{ch} \mu\xi - \alpha (\cos \mu\xi - \operatorname{ch} \mu\xi)$. One beam end built in, the other free (case 4)

$\mu_1 = 1.8751$ First term of expansion ($\alpha_1 = 1.362223$)					$\mu_2 = 4.6941$ Second term of expansion ($\alpha_2 = 0.981863$)				
$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_1} X'$	$\left(\frac{b}{\mu_1}\right)^2 X''$	$\left(\frac{b}{\mu_1}\right)^3 X'''$	$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_2} X'$	$\left(\frac{b}{\mu_2}\right)^2 X''$	$\left(\frac{b}{\mu_2}\right)^3 X'''$
0.0	0.0000	0.0000	2.7244	−2.0000	0.0	0.0000	0.0000	1.9637	−2.0000
0.1	0.4560	0.4747	2.3504	−1.9971	0.1	0.1816	0.7014	1.0293	−1.9703
0.2	0.1740	0.8813	1.9764	−1.9777	0.2	0.5914	0.9728	0.1372	−1.7936
0.3	0.3712	1.2665	1.6107	−1.9276	0.3	1.0330	0.8515	−0.6223	−1.4104
0.4	0.6262	1.4857	1.2563	−1.8347	0.4	1.3423	0.4227	−1.1579	−0.8443
0.5	0.9250	1.6898	0.9250	−1.6898	0.5	1.4014	−0.1895	−1.4014	−0.1895
0.6	1.2563	1.8347	0.6262	−1.4857	0.6	1.1579	−0.8443	−1.3423	0.4227
0.7	1.6107	1.9276	0.3712	−1.2665	0.7	0.6223	−1.4104	−1.0330	0.3515
0.8	1.9764	1.9777	0.1740	−0.8813	0.8	−0.1372	−1.7936	−0.5914	0.9723
0.9	2.3504	1.9971	0.0456	−0.4747	0.9	−1.0293	−1.9703	−0.1816	0.7014
1.0	2.7244	2.0000	0.0000	0.0000	1.0	−1.9637	−2.0000	0.0000	0.0000

$\mu_3 = 7.8548$ Third term of expansion ($\alpha_3 = 1.00078$)					$\mu_4 = 10.9956$ Fourth term of expansion ($\alpha_4 = 0.999967$)				
$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_3} X'$	$\left(\frac{b}{\mu_3}\right)^2 X''$	$\left(\frac{b}{\mu_3}\right)^3 X'''$	$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_4} X'$	$\left(\frac{b}{\mu_4}\right)^2 X''$	$\left(\frac{b}{\mu_4}\right)^3 X'''$
0.0	0.0000	0.0000	2.0016	−2.0000	0.0	0.0000	0.0000	1.9999	−2.0000
0.1	0.4560	0.9593	0.4583	−1.8702	0.1	0.7704	1.0119	−0.1048	−1.6777
0.2	1.2100	0.7945	−0.7904	−1.2066	0.2	1.5076	0.1090	−1.2863	−0.3309
0.3	1.5137	−0.0899	−1.3158	−0.0916	0.3	0.8661	−1.1826	−0.7933	1.1079
0.4	1.0547	−1.0327	−0.9501	0.9541	0.4	−0.6334	−1.2725	0.6552	1.2453
0.5	0.0395	−1.4147	0.0395	1.4147	0.5	−1.4141	−0.0081	1.4141	−0.0081
0.6	−0.9501	−0.9641	1.0547	1.0327	0.6	−0.6552	1.2453	0.6334	−1.2725
0.7	−1.3158	0.0916	1.5137	0.0899	0.7	0.7933	1.1079	−0.8661	−1.1826
0.8	−0.7904	1.2066	1.2100	−0.7945	0.8	1.2863	−0.3309	−1.5076	0.1090
0.9	0.4583	1.8702	0.4560	−0.9593	0.9	0.1048	−1.6777	−0.7704	1.0119
1.0	2.0016	2.0000	0.0000	0.0000	1.0	−1.9999	−2.0000	0.0000	0.0000

TABLE 9

$X(\xi) = \sin \mu\xi - \alpha \operatorname{ch} \mu\xi$. One beam end built in, the other simply supported (case 5)

$\mu_1 = 3.9266$
First term of expansion ($\alpha_1 = -0.0278749$)

$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_1} X'$	$\left(\frac{b}{\mu_1}\right)^2 X''$	$\left(\frac{b}{\mu_1}\right)^3 X'''$
0.0	0.0000	1.0279	0.0000	−0.9721
0.1	0.3939	0.9539	−0.3714	−0.8938
0.2	0.7310	0.7443	−0.6826	−0.6705
0.3	0.9648	0.4323	−0.8829	−0.3332
0.4	1.0642	0.0698	−0.9358	−0.0702
0.5	1.0214	−0.2810	−0.8268	0.4834
0.6	0.8529	−0.5586	−0.5615	0.8553
0.7	0.5995	−0.7052	−0.1657	1.1426
0.8	0.3230	−0.6774	0.3198	1.3226
0.9	0.0965	−0.4495	0.8539	1.4015
1.0	0.0000	0.0000	1.4136	1.4147

$\mu_2 = 7.0685$
Second term of expansion ($\alpha_2 = 0.00122285$)

$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_2} X'$	$\left(\frac{b}{\mu_2}\right)^2 X''$	$\left(\frac{b}{\mu_2}\right)^3 X'''$
0.0	0.0000	0.9988	0.0000	−1.0012
0.1	0.6496	0.7588	−0.6505	−0.7618
0.2	0.9854	0.1535	−0.9901	−0.1588
0.3	0.8475	−0.5279	−0.8574	0.5178
0.4	0.2991	−0.9613	−0.3197	0.9406
0.5	−0.3996	−0.9464	0.3579	0.9046
0.6	−0.9323	−0.4968	0.8487	0.4132
0.7	−1.0569	0.1504	0.8869	−0.3204
0.8	−0.7555	0.6323	0.4179	−0.9800
0.9	−0.2714	0.6489	−0.4249	−1.3452
1.0	0.0000	0.0000	−1.4141	−1.4143

$\mu_3 = 10.2102$
Third term of expansion ($\alpha_3 = 0.0000520346$)

$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_3} X'$	$\left(\frac{b}{\mu_3}\right)^2 X''$	$\left(\frac{b}{\mu_3}\right)^3 X'''$
0.0	0.0000	1.0000	0.0000	−1.0000
0.1	0.8527	0.5226	−0.8525	−0.5224
0.2	0.8912	−0.4537	−0.8908	0.4542
0.3	0.0821	−0.9961	−0.0809	0.9972
0.4	−0.8051	−0.5895	0.8082	0.5926
0.5	−0.9193	0.3869	0.9279	−0.3783
0.6	−0.1406	1.0002	0.1645	−0.9763
0.7	0.7954	0.6804	−0.7291	−0.6141
0.8	1.0424	−0.2188	−0.8586	0.4027
0.9	0.4876	−0.7176	0.0223	1.2275
1.0	0.0000	0.0000	1.4142	1.4142

$\mu_4 = 13.3520$
Fourth term of expansion ($\alpha_4 = 0.00000224861$)

$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_4} X'$	$\left(\frac{b}{\mu_4}\right)^2 X''$	$\left(\frac{b}{\mu_4}\right)^3 X'''$
0.0	0.0000	1.0000	0.0000	−1.0000
0.1	0.9723	0.2336	−0.9723	−0.2336
0.2	0.4543	−0.8908	−0.4543	0.8908
0.3	−0.7634	−0.6461	0.7632	0.6459
0.4	−0.8096	0.5870	0.8092	−0.5874
0.5	0.3856	0.9214	−0.3874	−0.9232
0.6	0.9845	−0.1588	−0.9912	0.1520
0.7	0.0618	−1.0101	−0.0876	0.9843
0.8	−1.0004	−0.3601	0.9009	0.2606
0.9	−0.7064	0.6676	0.3328	−1.0412
1.0	0.0000	0.0000	−1.4145	−1.4139

TABLE 10

$X(\xi) = \sin \mu\xi + \alpha \operatorname{sh} \mu\xi$. One beam end simply supported, the other free (case 6)

$\mu_1 = 3.9266$ First term of expansion ($\alpha_1 = 0.0278749$)					$\mu_2 = 7.0685$ Second term of expansion ($\alpha_2 = 0.00122295$)				
$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_1} X'$	$\left(\frac{b}{\mu_1}\right)^2 X''$	$\left(\frac{b}{\mu_1}\right)^3 X'''$	$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_2} X'$	$\left(\frac{b}{\mu_2}\right)^2 X''$	$\left(\frac{b}{\mu_2}\right)^3 X'''$
0.0	0.0000	0.9721	0.0000	−1.0279	0.0	0.0000	1.0012	0.0000	−0.9988
0.1	0.3714	0.8938	−0.3939	−0.9539	0.1	0.6505	0.7618	−0.6486	−0.7588
0.2	0.6826	0.6705	−0.7310	−0.7443	0.2	0.9901	0.1588	−0.9854	−0.1535
0.3	0.8829	0.3332	−0.9648	−0.4323	0.3	0.8574	−0.5178	−0.8475	0.5279
0.4	0.9358	−0.0702	−1.0642	−0.0698	0.4	0.3197	−0.9406	−0.2991	0.9613
0.5	0.8268	−0.4834	−1.0214	0.2810	0.5	−0.3579	−0.9046	0.3996	0.9464
0.6	0.5615	−0.8553	−0.8529	0.5586	0.6	−0.8487	−0.4132	0.9323	0.4968
0.7	0.1657	−1.1426	−0.5995	0.7052	0.7	−0.8869	0.3204	1.0569	−0.1504
0.8	−0.3198	−1.3226	−0.3230	0.6774	0.8	−0.4179	0.9800	0.7555	−0.6323
0.9	−0.8539	−1.4015	−0.0965	0.4495	0.9	0.4249	1.3452	0.2714	−0.6489
1.0	−1.4136	−1.4147	0.0000	0.0000	1.0	1.4141	1.4143	0.0000	0.0000

$\mu_3 = 10.2102$ Third term of expansion ($\alpha_3 = -0.0000520346$)					$\mu_4 = 13.3520$ Fourth term of expansion ($\alpha_4 = 0.00000224861$)				
$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_3} X'$	$\left(\frac{b}{\mu_3}\right)^2 X''$	$\left(\frac{b}{\mu_3}\right)^3 X'''$	$\xi = \frac{x}{b}$	X	$\frac{b}{\mu_4} X'$	$\left(\frac{b}{\mu_4}\right)^2 X''$	$\left(\frac{b}{\mu_4}\right)^3 X'''$
0.0	0.0000	1.0000	0.0000	−1.0000	0.0	0.0000	1.0000	0.0000	−1.0000
0.1	0.8525	−0.5224	−0.8527	−0.5226	0.1	0.9723	0.2336	−0.9723	−0.2336
0.2	0.8908	−0.4542	−0.8912	0.4537	0.2	0.4543	−0.8908	−0.4543	0.8908
0.3	0.0809	−0.9272	−0.0821	0.9961	0.3	−0.7632	−0.6459	0.7634	0.6461
0.4	−0.8082	−0.5926	0.8051	0.5895	0.4	−0.8092	0.5874	0.8096	−0.5870
0.5	−0.9279	0.3783	0.9193	−0.3869	0.5	0.3874	0.9232	−0.3856	−0.9214
0.6	−0.1645	0.9763	0.1406	−1.0002	0.6	0.9912	−0.1520	−0.9845	0.1588
0.7	0.7291	0.6141	−0.7954	−0.6804	0.7	0.0876	−0.9843	−0.0618	1.0101
0.8	0.8586	−0.4027	−1.0424	0.2188	0.8	−0.9009	−0.2606	1.0004	0.2601
0.9	−0.0223	−1.2275	−0.4876	0.7176	0.9	−0.3328	1.0412	0.7064	−0.6676
1.0	−1.4142	−1.4142	0.0000	0.0000	1.0	1.4145	1.4139	0.0000	0.0000

TABLE 11

Thomson functions

ξ	$u_0(\xi)$	$v_0(\xi)$	$f_0(\xi)$	$g_0(\xi)$	ξ	$u_0(\xi)$	$v_0(\xi)$	$f_0(\xi)$	$g_0(\xi)$
0.00	1.0000	−0.0000	0.5000	$-\infty$	0.82	0.9929	−0.1680	0.3560	−0.2760
0.02	1.0000	−0.0001	0.4997	−2.5643	0.84	0.9922	−0.1762	0.3514	−0.2641
0.04	1.0000	−0.0004	0.4989	−2.1232	0.86	0.9915	−0.1847	0.3469	−0.2526
0.06	1.0000	−0.0009	0.4978	−1.8653	0.88	0.9906	−0.1934	0.3423	−0.2415
0.08	1.0000	−0.0016	0.4963	−1.6825	0.90	0.9898	−0.2023	0.3377	−0.2308
0.10	1.0000	−0.0025	0.4946	−1.5409	0.92	0.9888	−0.2113	0.3332	−0.2205
0.12	1.0000	−0.0036	0.4926	−1.4254	0.94	0.9878	−0.2206	0.3286	−0.2105
0.14	1.0000	−0.0049	0.4904	−1.3279	0.96	0.9867	−0.2301	0.3241	−0.2008
0.16	1.0000	−0.0064	0.4880	−1.2436	0.98	0.9856	−0.2397	0.3196	−0.1915
0.18	1.0000	−0.0081	0.4854	−1.1695	1.00	0.9844	−0.2496	0.3151	−0.1825
0.20	1.0000	−0.0100	0.4826	−1.1033	1.10	0.9771	−0.3017	0.2929	−0.1419
0.22	1.0000	−0.0121	0.4797	−1.0437	1.20	0.9676	−0.3587	0.2713	−0.1076
0.24	0.9999	−0.0144	0.4767	−0.9894	1.30	0.9554	−0.4204	0.2504	−0.0786
0.26	0.9999	−0.0169	0.4735	−0.9397	1.40	0.9401	−0.4867	0.2302	−0.0542
0.28	0.9999	−0.0196	0.4701	−0.8938	1.50	0.9211	−0.5576	0.2110	−0.0337
0.30	0.9999	−0.0225	0.4667	−0.8513	1.60	0.8979	−0.6327	0.1926	−0.0166
0.32	0.9998	−0.0256	0.4632	−0.8117	1.70	0.8700	−0.7120	0.1752	−0.0023
0.34	0.9998	−0.0289	0.4595	−0.7747	1.80	0.8367	−0.7953	0.1588	0.0094
0.36	0.9997	−0.0324	0.4558	−0.7400	1.90	0.7975	−0.8821	0.1433	0.0189
0.38	0.9997	−0.0361	0.4520	−0.7073	2.00	0.7517	−0.9723	0.1289	0.0265
0.40	0.9996	−0.0400	0.4480	−0.6765	2.20	0.6377	−1.1610	0.1026	0.0371
0.42	0.9995	−0.0441	0.4441	−0.6473	2.4	0.4890	−1.3575	0.0804	0.0429
0.44	0.9994	−0.0484	0.4400	−0.6198	2.6	0.3001	−1.5569	0.0614	0.0446
0.46	0.9993	−0.0529	0.4359	−0.5935	2.8	0.0651	−1.7529	0.0455	0.0447
0.48	0.9992	−0.0576	0.4318	−0.5686	3.0	−0.2214	−1.9376	0.0326	0.0427
0.50	0.9990	−0.0625	0.4275	−0.5449	3.2	−0.5644	−2.1016	0.0220	0.0394
0.52	0.9989	−0.0676	0.4233	−0.5223	3.4	−0.9680	−2.2334	0.0137	0.0356
0.54	0.9987	−0.0729	0.4190	−0.5006	3.6	−1.4353	−2.3199	0.0072	0.0314
0.56	0.9985	−0.0784	0.4146	−0.4800	3.8	−1.9674	−2.3454	0.0022	0.0260
0.58	0.9982	−0.0841	0.4102	−0.4602	4.0	−2.5634	−2.2927	−0.0014	0.0230
0.60	0.9980	−0.0900	0.4058	−0.4413	4.2	−3.2195	−2.1422	−0.0039	0.0192
0.62	0.9977	−0.0961	0.4014	−0.4231	4.4	−3.9283	−1.8726	−0.0056	0.0156
0.64	0.9974	−0.1024	0.3969	−0.4057	4.6	−4.6784	−1.4610	−0.0066	0.0125
0.66	0.9970	−0.1089	0.3924	−0.3889	4.8	−5.4531	−0.8837	−0.0071	0.0097
0.68	0.9967	−0.1156	0.3879	−0.3729	5.0	−6.2301	−0.1160	−0.0071	0.0073
0.70	0.9962	−0.1224	0.3834	−0.3574	5.2	−6.9803	0.8658	−0.0069	0.0053
0.72	0.9958	−0.1295	0.3788	−0.3425	5.4	−7.6674	2.0845	−0.0065	0.0037
0.74	0.9953	−0.1368	0.3743	−0.3282	5.6	−8.2466	3.5597	−0.0059	0.0023
0.76	0.9948	−0.1443	0.3697	−0.3144	5.8	−8.6644	5.3068	−0.0053	0.0012
0.78	0.9942	−0.1520	0.3651	−0.3011	6.0	−8.8583	7.3347	−0.0046	0.0004
0.80	0.9936	−0.1599	0.3606	−0.2883					

TABLE 12

Kinematic and force factors for a circular plate

ξ	$u_0(\xi)$	$v_0(\xi)$	$\theta_1(\xi)$	$\theta_2(\xi)$	$M_1(\xi)$	$M_2(\xi)$	$\overline{M}_1(\xi)$	$\overline{M}_2(\xi)$	$Q_1(\xi)$	$Q_2(\xi)$
					$\varphi = 46°$					
0.00	1.0000	0.0000	0.0000	0.0000	−0.0349	0.9994	−0.0174	0.4947	0.0000	0.0000
0.05	1.0000	−0.0006	−0.0009	0.0250	−0.0343	0.9995	−0.0172	0.4997	−0.0249	−0.0017
0.10	1.0000	−0.0024	−0.0016	0.0500	−0.0324	0.9996	−0.0168	0.4997	−0.0499	−0.0034
0.15	1.0002	−0.0056	−0.0024	0.0749	−0.0293	0.9998	−0.0160	0.4998	−0.0748	−0.0050
0.20	1.0003	−0.0100	−0.0030	0.0999	−0.0249	1.0001	−0.0149	0.4999	−0.0998	−0.0065
0.25	1.0005	−0.0156	−0.0034	0.1249	−0.0193	1.0004	−0.0155	0.5000	−0.1248	−0.0077
0.30	1.0007	−0.0225	−0.0035	0.1500	−0.0125	1.0009	−0.0118	0.5001	−0.1498	−0.0088
0.35	1.0009	−0.0306	−0.0034	0.1751	−0.0044	1.0013	−0.0098	0.5002	−0.1748	−0.0095
0.40	1.0010	−0.0400	−0.0030	0.2002	0.0050	1.0018	−0.0074	0.5003	−0.1999	−0.0100
0.45	1.0012	−0.0506	−0.0021	0.2253	0.0157	1.0023	−0.0048	0.5005	−0.2250	−0.0100
0.50	1.0012	−0.0626	−0.0009	0.2504	0.0275	1.0029	−0.0018	0.5006	−0.2501	−0.0097
0.55	1.0012	−0.0770	0.0008	0.2754	0.0406	1.0033	0.0015	0.5008	−0.2753	−0.0089
0.60	1.0011	−0.0900	0.0030	0.3005	0.0551	1.0037	0.0050	0.5009	−0.3004	−0.0065
0.65	1.0011	−0.1058	0.0058	0.3257	0.0708	1.0040	0.0089	0.5010	−0.3257	−0.0056
0.70	1.0006	−0.1227	0.0092	0.3509	0.0877	1.0042	0.0133	0.5012	−0.3510	−0.0030
0.75	1.0000	−0.1404	0.0132	0.3760	0.1058	1.0043	0.0178	0.5013	−0.3762	0.0001
0.80	0.9992	−0.1602	0.0180	0.4012	0.1253	1.0044	0.0227	0.5014	−0.4015	0.0040
0.85	0.9982	−0.1810	0.0236	0.4263	0.1459	1.0039	0.0277	0.5015	−0.4269	0.0089
0.90	0.9969	−0.2029	0.0300	0.4514	0.1680	1.0033	0.0333	0.5015	−0.4522	0.0141
0.95	0.9951	−0.2261	0.0371	0.4764	0.1912	1.0025	0.0391	0.5015	−0.4778	0.0204
1.00	0.9932	−0.2504	0.0453	0.5015	0.2151	1.0012	0.0452	0.5015	−0.4923	0.0276
					$\varphi = 47°$					
0.00	1.0000	0.0000	0.0000	0.0000	−0.0698	0.9976	−0.0349	0.4988	0.0000	0.0000
0.05	1.0000	0.0006	−0.0017	0.0249	−0.0692	0.9977	−0.0347	0.4988	−0.0247	−0.0035
0.10	1.0001	0.0024	−0.0034	0.0499	−0.0673	0.9979	−0.0343	0.4989	−0.0495	−0.0069
0.15	1.0004	0.0056	−0.0050	0.0748	−0.0642	0.9984	−0.0335	0.4990	−0.0743	−0.0102
0.20	1.0007	0.0100	−0.0065	0.0999	−0.0599	0.9990	−0.0324	0.4991	−0.0991	−0.0134
0.25	1.0011	0.0156	−0.0077	0.1248	−0.0543	0.9997	−0.0310	0.4993	−0.1240	−0.0165
0.30	1.0016	0.0224	−0.0088	0.1498	−0.0475	1.0006	−0.0293	0.4996	−0.1488	−0.0193
0.35	1.0021	0.0305	−0.0096	0.1750	−0.0395	1.0017	−0.0273	0.4999	−0.1738	−0.0218
0.40	1.0026	0.0399	−0.0100	0.2000	−0.0301	1.0028	−0.0250	0.5001	−0.1989	−0.0239
0.45	1.0029	0.0506	−0.0101	0.2253	−0.0196	1.0040	−0.0224	0.5005	−0.2239	−0.0257
0.50	1.0034	0.0624	−0.0098	0.2504	−0.0077	1.0053	−0.0194	0.5008	−0.2490	−0.0272
0.55	1.0039	0.0756	−0.0089	0.2756	0.0054	1.0067	−0.0162	0.5012	−0.2744	−0.0281
0.60	1.0043	0.0901	−0.0075	0.3010	0.0197	1.0081	−0.0127	0.5016	−0.2997	−0.0286
0.65	1.0048	0.1058	−0.0056	0.3263	0.0354	1.0095	−0.0087	0.5020	−0.3251	−0.0283
0.70	1.0048	0.1227	−0.0031	0.3518	0.0523	1.0109	−0.0045	0.5025	−0.3506	−0.0276
0.75	1.0049	0.1409	0.0000	0.3772	0.0703	1.0124	0.0001	0.5029	−0.3763	−0.0262
0.80	1.0049	0.1604	0.0040	0.4027	0.0898	1.0136	0.0049	0.5034	−0.4019	−0.0242
0.85	1.0045	0.1811	0.0087	0.4283	0.1107	1.0147	0.0101	0.5038	−0.4276	−0.0213
0.90	1.0040	0.2032	0.0140	0.4538	0.1326	1.0157	0.0156	0.5041	−0.4537	−0.0156
0.95	1.0030	0.2265	0.0203	0.4796	0.1559	1.0165	0.0214	0.5045	−0.4796	−0.0131
1.00	1.0018	0.2511	0.0275	0.5052	0.1810	1.0170	0.0265	0.5050	−0.5056	−0.0078
					$\varphi = 48°$					
0.00	1.0000	0.0000	0.0000	0.0000	−0.1045	0.9945	−0.0522	0.4977	0.0000	0.0000
0.05	1.0000	0.0006	−0.0026	0.0249	−0.1039	0.9946	−0.0520	0.4977	−0.0244	−0.0052
0.10	1.0002	0.0024	−0.0051	0.0497	−0.1021	0.9950	−0.0516	0.4978	−0.0489	−0.0103
0.15	1.0006	0.0506	−0.0076	0.0746	−0.0990	0.9957	−0.0508	0.4980	−0.0734	−0.0154
0.20	1.0010	0.0099	−0.0100	0.0996	−0.0947	0.9966	−0.0498	0.4982	−0.0979	−0.0203
0.25	1.0016	0.0155	−0.0122	0.1245	−0.0892	0.9977	−0.0484	0.4985	−0.1225	−0.0251
0.30	1.0022	0.0224	−0.0139	0.1495	−0.0825	0.9991	−0.0467	0.4989	−0.1472	−0.0296
0.35	1.0030	0.0304	−0.0157	0.1746	−0.0745	1.0007	−0.0447	0.4993	−0.1720	−0.0339
0.40	1.0038	0.0399	−0.0170	0.1997	−0.0653	1.0024	−0.0424	0.4997	−0.1968	−0.0378
0.45	1.0047	0.0504	−0.0179	0.2250	−0.0548	1.0044	−0.0398	0.5001	−0.2218	−0.0414

ξ	$u_0(\xi)$	$v_0(\xi)$	$\theta_1(\xi)$	$\theta_2(\xi)$	$M_1(\xi)$	$M_2(\xi)$	$\overline{M}_1(\xi)$	$\overline{M}_2(\xi)$	$Q_1(\xi)$	$Q_2(\xi)$
0.50	1.0056	0.0623	−0.0185	0.2501	−0.0431	1.0066	−0.0369	0.5007	−0.2468	−0.0446
0.55	1.0065	0.0755	−0.0185	0.2756	−0.0301	1.0089	−0.0336	0.5014	−0.2721	−0.0473
0.60	1.0074	0.0899	−0.0182	0.3009	−0.0159	1.0113	−0.0301	0.5021	−0.2975	−0.0495
0.65	1.0083	0.1056	−0.0171	0.3265	−0.0003	1.0148	−0.0263	0.5028	−0.3229	−0.0512
0.70	1.0091	0.1226	−0.0155	0.3522	0.0164	1.0164	−0.0220	0.5035	−0.3485	−0.0522
0.75	1.0099	0.1407	−0.0132	0.3778	0.0344	1.0190	−0.0176	0.5042	−0.3743	−0.0527
0.80	1.0105	0.1603	−0.0102	0.4036	0.0539	1.0217	−0.0128	0.5050	−0.4003	−0.0524
0.85	1.0109	0.1812	−0.0064	0.4229	0.0746	1.0242	−0.0076	0.5058	−0.4265	−0.0514
0.90	1.0111	0.2033	0.0001	0.4555	0.0966	1.0268	−0.0022	0.5065	−0.4528	−0.0496
0.95	1.0111	0.2267	0.0035	0.4876	0.1198	1.0293	0.0006	0.5074	−0.4793	−0.0470
1.00	1.0107	0.2514	0.0097	0.5077	0.1444	1.0314	0.0097	0.5082	−0.5059	−0.0435
					$\varphi = 49°$					
0.00	1.0000	0.0000	0.0000	0.0000	−0.1392	0.9903	−0.0696	0.4951	0.0000	0.0000
0.05	1.0000	−0.0006	−0.0035	0.0248	−0.1386	0.9905	−0.0695	0.4951	−0.0240	−0.0068
0.10	1.0003	−0.0024	−0.0069	0.0495	−0.1368	0.9910	−0.0690	0.4953	−0.0480	−0.0138
0.15	1.0007	−0.0056	−0.0102	0.0743	−0.1338	0.9918	−0.0682	0.4955	−0.0722	−0.0205
0.20	1.0013	−0.0099	−0.0134	0.0991	−0.1296	0.9930	−0.0672	0.4958	−0.0963	−0.0272
0.25	1.0021	−0.0155	−0.0165	0.1241	−0.1242	0.9946	−0.0658	0.4962	−0.1206	−0.0337
0.30	1.0028	−0.0223	−0.0198	0.1491	−0.1176	0.9964	−0.0642	0.4966	−0.1449	−0.0338
0.35	1.0038	−0.0304	−0.0218	0.1740	−0.1079	0.9985	−0.0622	0.4972	−0.1693	−0.0468
0.40	1.0049	−0.0397	−0.0240	0.1992	−0.1006	1.0009	−0.0600	0.4977	−0.1935	−0.0515
0.45	1.0061	−0.0503	−0.0258	0.2245	−0.0902	1.0036	−0.0574	0.4985	−0.2186	−0.0568
0.50	1.0073	−0.0622	−0.0273	0.2496	−0.0787	1.0066	−0.0545	0.4993	−0.2434	−0.0618
0.55	1.0087	−0.0753	−0.0283	0.2752	−0.0656	1.0098	−0.0513	0.5001	−0.2684	−0.0662
0.60	1.0100	−0.0896	−0.0287	0.3006	−0.0519	1.0133	−0.0479	0.5010	−0.2937	−0.0703
0.65	1.0112	−0.1054	−0.0286	0.3263	−0.0366	1.0169	−0.0440	0.5020	−0.3192	−0.0737
0.70	1.0125	−0.1213	−0.0279	0.3552	−0.0200	1.0207	−0.0400	0.5029	−0.3447	−0.0767
0.75	1.0138	−0.1406	−0.0267	0.3781	−0.0020	1.0245	−0.0355	0.5040	−0.3707	−0.0790
0.80	1.0149	−0.1602	−0.0246	0.4042	0.0171	1.0285	−0.0308	0.5051	−0.3968	−0.0805
0.85	1.0160	−0.1810	−0.0219	0.4306	0.0376	1.0326	−0.0257	0.5063	−0.4231	−0.0814
0.90	1.0168	−0.2032	−0.0182	0.4568	0.0594	1.0367	−0.0202	0.5074	−0.4498	−0.0815
0.95	1.0174	−0.2266	−0.0140	0.4835	0.0826	1.0407	−0.0145	0.5087	−0.4757	−0.0809
1.00	1.0178	−0.2515	−0.0085	0.5100	0.1070	1.0447	−0.0084	0.5099	−0.5038	−0.0795
					$\varphi = 50°$					
0.00	1.0000	0.0000	0.0000	0.0000	−0.1736	0.9848	−0.0868	0.4924	0.0000	0.0000
0.05	1.0001	−0.0006	−0.0043	0.0245	−0.1730	0.9850	−0.0867	0.4924	−0.0233	−0.0085
0.10	1.0004	−0.0024	−0.0087	0.0491	−0.1713	0.9856	−0.0862	0.4926	−0.0469	−0.0170
0.15	1.0010	−0.0055	−0.0129	0.0738	−0.1683	0.9866	−0.0855	0.4929	−0.0705	−0.0255
0.20	1.0017	−0.0098	−0.0168	0.0986	−0.1643	0.9881	−0.0845	0.4932	−0.0941	−0.0338
0.25	1.0026	−0.0154	−0.0208	0.1233	−0.1588	0.9829	−0.0831	0.4937	−0.1178	−0.0418
0.30	1.0038	−0.0222	−0.0248	0.1485	−0.1524	0.9922	−0.0815	0.4943	−0.1419	−0.0502
0.35	1.0051	−0.0302	−0.0278	0.1731	−0.1448	0.9994	−0.0796	0.4950	−0.1658	−0.0575
0.40	1.0066	−0.0395	−0.0309	0.1939	−0.1358	0.9980	−0.0774	0.4957	−0.1899	−0.0648
0.45	1.0082	−0.0501	−0.0336	0.2241	−0.1258	1.0014	−0.0749	0.4966	−0.2141	−0.0719
0.50	1.0099	−0.0619	−0.0360	0.2494	−0.1144	1.0052	−0.0721	0.4976	−0.2386	−0.0783
0.55	1.0118	−0.0749	−0.0379	0.2748	−0.1019	1.0093	−0.0690	0.4987	−0.2633	−0.0849
0.60	1.0137	−0.0893	−0.0392	0.3004	−0.0881	1.0138	−0.0655	0.4999	−0.2883	−0.0907
0.65	1.0157	−0.1049	−0.0399	0.3262	−0.0730	1.0184	−0.0618	0.5010	−0.3135	−0.0960
0.70	1.0177	−0.1218	−0.0404	0.3521	−0.0567	1.0228	−0.0577	0.5024	−0.3390	−0.1008
0.75	1.0197	−0.1401	−0.0402	0.3784	−0.0391	1.0285	−0.0534	0.5037	−0.3645	−0.1050
0.80	1.0217	−0.1596	−0.0389	0.4047	−0.0203	1.0338	−0.0487	0.5052	−0.39[illegible]1	−0.1071
0.85	1.0236	−0.1805	−0.0371	0.4306		1.0393	−0.0437	0.5066	−0.4176	−0.1113
0.90	1.0254	−0.2027	−0.0345	0.4572		1.0450	−0.0385	0.5082	−0.4443	−0.1135
0.95	1.0271	−0.2263	−0.0311	0.4843		1.0507	−0.0328	0.5099	−0.4714	−0.1147
1.00	1.0285	−0.2511	−0.0267	0.5114		1.0565	−0.0268	0.5116	−0.4990	−0.1153
					$\varphi = 51°$					
0.00	1.0000	0.0000	0.0000	0.0000	−0.2079	0.9781	−0.1039	0.4890	−0.0000	0.0000
0.05	1.0001	0.0006	−0.0052	0.0244	−0.2073	0.9783	−0.1038	0.4891	−0.0228	−0.0102
0.10	1.0005	0.0024	−0.0104	0.0489	−0.2056	0.9791	−0.1033	0.4892	−0.0457	−0.0203
0.15	1.0012	0.0055	−0.0154	0.0733	−0.2028	0.9804	−0.1026	0.4896	−0.0686	−0.0303
0.20	1.0021	0.0098	−0.0204	0.0980	−0.1988	0.9832	−0.1016	0.4900	−0.0916	−0.0405
0.25	1.0032	0.0153	−0.0259	0.1227	−0.1936	0.9844	−0.1003	0.4906	−0.1147	−0.0500
0.30	1.0046	0.0220	−0.0297	0.1474	−0.1873	0.9871	−0.0987	0.4913	−0.1380	−0.0596

ξ	$u_0(\xi)$	$v_0(\xi)$	$\theta_1(\xi)$	$\theta_2(\xi)$	$M_1(\xi)$	$M_2(\xi)$	$\overline{M}_1(\xi)$	$\overline{M}_2(\xi)$	$Q_1(\xi)$	$Q_2(\xi)$
0.35	1.0062	0.0300	−0.0340	0.1723	−0.1798	0.9913	−0.0969	0.4921	−0.1614	−0.0690
0.40	1.0080	0.0393	−0.0380	0.1972	−0.1712	0.9947	−0.0948	0.4931	−0.1850	−0.0776
0.45	1.0100	0.0498	−0.0416	0.2224	−0.1613	0.9992	−0.0922	0.4940	−0.2088	−0.0869
0.50	1.0121	0.0615	−0.0449	0.2476	−0.1502	1.0027	−0.0895	0.4952	−0.2330	−0.1008
0.55	1.0144	0.0745	−0.0476	0.2731	−0.1380	1.0087	−0.0865	0.4965	−0.2572	−0.1033
0.60	1.0169	0.0888	−0.0500	0.2987	−0.1245	1.0141	−0.0832	0.4978	−0.2818	−0.1109
0.65	1.0194	0.1044	−0.0517	0.3247	−0.1098	1.0188	−0.0795	0.4993	−0.3069	−0.1181
0.70	1.0221	0.1213	−0.0529	0.3507	−0.0938	1.0259	−0.0755	0.5009	−0.3320	−0.1247
0.75	1.0247	0.1394	−0.0535	0.3770	−0.0766	1.0322	−0.0713	0.5026	−0.3577	−0.1308
0.80	1.0274	0.1590	−0.0534	0.4035	−0.0581	1.0383	−0.0668	0.5044	−0.3836	−0.1362
0.85	1.0300	0.1799	−0.0526	0.4304	−0.0382	1.0448	−0.0619	0.5063	−0.4100	−0.1410
0.90	1.0326	0.2021	−0.0509	0.4509	−0.0170	1.0519	−0.0567	0.5082	−0.4369	−0.1451
0.95	1.0351	0.2256	−0.0486	0.4848	−0.0055	1.0593	−0.0536	0.5102	−0.4641	−0.1484
1.00	1.0433	0.2505	−0.0454	0.5123	−0.0004	1.0668	−0.0453	0.5123	−0.4917	−0.1509
					$\varphi = 52°$					
0.00	1.0000	0.0000	0.0000	0.0000	−0.2419	0.9703	−0.1209	0.4851	0.0000	0.0000
0.05	1.0001	0.0006	−0.0060	0.0242	−0.2414	0.9705	−0.1208	0.4852	−0.0200	−0.0117
0.10	1.0006	0.0024	−0.0120	0.0485	−0.2398	0.9714	−0.1204	0.4854	−0.0441	−0.0235
0.15	1.0013	0.0054	−0.0179	0.0729	−0.2369	0.9729	−0.1197	0.4858	−0.0601	−0.0350
0.20	1.0024	0.0153	−0.0238	0.0972	−0.2331	0.9753	−0.1187	0.4863	−0.0886	−0.0466
0.25	1.0038	0.0207	−0.0293	0.1218	−0.2281	0.9775	−0.1175	0.4869	−0.1106	−0.0580
0.30	1.0053	0.0218	−0.0348	0.1463	−0.2219	0.9808	−0.1159	0.4877	−0.1335	−0.0692
0.35	1.0072	0.0298	−0.0399	0.1713	−0.2148	0.9845	−0.1142	0.4887	−0.1518	−0.0802
0.40	1.0093	0.0400	−0.0449	0.1960	−0.2063	0.9888	−0.1121	0.4898	−0.1793	−0.0909
0.45	1.0117	0.0494	−0.0494	0.2210	−0.1968	0.9936	−0.1096	0.4909	−0.2000	−0.1014
0.50	1.0142	0.0610	−0.0536	0.2463	−0.1861	0.9989	−0.1070	0.4923	−0.2259	−0.1115
0.55	1.0169	0.0742	−0.0572	0.2716	−0.1742	1.0051	−0.1040	0.4938	−0.2468	−0.1213
0.60	1.0199	0.0892	−0.0606	0.2973	−0.1612	1.0110	−0.1008	0.4955	−0.2738	−0.1307
0.65	1.0229	0.1038	−0.0633	0.3231	−0.1468	1.0178	−0.0973	0.4972	−0.3013	−0.1369
0.70	1.0262	0.1206	−0.0654	0.3494	−0.1313	1.0250	−0.0935	0.4990	−0.3231	−0.1480
0.75	1.0294	0.1387	−0.0670	0.3758	−0.1145	1.0325	−0.0894	0.5010	−0.3473	−0.1560
0.80	1.0328	0.1582	−0.0680	0.4025	−0.0964	1.0406	−0.0849	0.5031	−0.3741	−0.1633
0.85	1.0362	0.1790	−0.0671	0.4295	−0.0771	1.0489	−0.0801	0.5053	−0.3951	−0.1699
0.90	1.0395	0.2011	−0.0677	0.4569	−0.0464	1.0576	−0.0751	0.5076	−0.4270	−0.1761
0.95	1.0429	0.2247	−0.0663	0.4845	−0.0344	1.0663	−0.0697	0.5100	−0.4547	−0.1814
1.00	1.0477	0.2496	−0.0640	0.5125	−0.0110	1.0756	−0.0640	0.5125	−0.4818	−0.1861
					$\varphi = 53°$					
0.00	1.0000	0.0000	0.0000	0.0000	−0.2756	0.9613	−0.1378	0.4806	0.0000	0.0000
0.05	1.0002	0.0006	−0.0069	0.0240	−0.2751	0.9616	−0.1377	0.4807	−0.0200	−0.0132
0.10	1.0007	0.0024	−0.0138	0.0481	−0.2735	0.9626	−0.1373	0.4809	−0.0424	−0.0265
0.15	1.0015	0.0054	−0.0205	0.0722	−0.2708	0.9643	−0.1366	0.4813	−0.0602	−0.0396
0.20	1.0027	0.0096	−0.0272	0.0964	−0.2671	0.9666	−0.1357	0.4819	−0.0852	−0.0527
0.25	1.0043	0.0150	−0.0336	0.1207	−0.2623	0.9696	−0.1345	0.4827	−0.1107	−0.0656
0.30	1.0061	0.0216	−0.0399	0.1451	−0.2564	0.9731	−0.1330	0.4836	−0.1284	−0.0784
0.35	1.0082	0.0295	−0.0459	0.1696	−0.2496	0.9774	−0.1313	0.4846	−0.1520	−0.0909
0.40	1.0107	0.0386	−0.0517	0.1944	−0.2414	0.9822	−0.1293	0.4859	−0.1726	−0.1033
0.45	1.0134	0.0490	−0.0572	0.2193	−0.2322	0.9877	−0.1270	0.4872	−0.1942	−0.1154
0.50	1.0164	0.0606	−0.0622	0.2444	−0.2219	0.9938	−0.1245	0.4888	−0.2178	−0.1272
0.55	1.0196	0.0735	−0.0669	0.2697	−0.2142	1.0005	−0.1216	0.4904	−0.2377	−0.1387
0.60	1.0213	0.0876	−0.0712	0.2954	−0.1978	1.0077	−0.1185	0.4923	−0.2643	−0.1498
0.65	1.0267	0.1030	−0.0749	0.3213	−0.1839	1.0153	−0.1151	0.4943	−0.2826	−0.1604
0.70	1.0305	0.1197	−0.0790	0.3475	−0.1689	1.0237	−0.1113	0.4965	−0.3124	−0.1708
0.75	1.0344	0.1378	−0.0804	0.3741	−0.1527	1.0324	−0.1074	0.4987	−0.3393	−0.1805
0.80	1.0385	0.1571	−0.0824	0.4008	−0.1352	1.0417	−0.1031	0.5011	−0.3626	−0.1898
0.85	1.0427	0.1778	−0.0837	0.4100	−0.1164	1.0515	−0.0986	0.5036	−0.3881	−0.1985
0.90	1.0468	0.1998	−0.0843	0.4557	−0.0964	1.0616	−0.0936	0.5063	−0.4148	−0.2066
0.95	1.0511	0.2234	−0.0840	0.4857	−0.0748	1.0720	−0.0884	0.5091	−0.4390	−0.2141
1.00	1.0553	0.2482	−0.0829	0.5120	−0.0522	1.0829	−0.0829	0.5120	−0.4693	−0.2208
					$\varphi = 54°$					
0.00	1.0000	0.0000	0.0000	0.0000	−0.3090	0.9511	−0.1545	0.4755	0.0000	0.0000
0.05	1.0002	0.0006	−0.0077	0.0238	−0.3085	0.9515	−0.1544	0.4756	−0.0202	−0.0149
0.10	1.0008	0.0023	−0.0154	0.0475	−0.3070	0.9525	−0.1540	0.4759	−0.0404	−0.0299
0.15	1.0017	0.0053	−0.0230	0.0714	−0.3045	0.9544	−0.1534	0.4763	−0.0609	−0.0447
0.20	1.0031	0.0095	−0.0305	0.0954	−0.3009	0.9570	−0.1525	0.4770	−0.0813	−0.0595
0.25	1.0048	0.0148	−0.0378	0.1195	−0.2964	0.9603	−0.1513	0.4778	−0.1019	−0.0741
0.30	1.0068	0.0215	−0.0449	0.1437	−0.2907	0.9642	−0.1500	0.4788	−0.1227	−0.0887
0.35	1.0093	0.0292	−0.0519	0.1680	−0.2840	0.9690	−0.1483	0.4800	−0.1438	−0.1030

ξ	$u_0(\xi)$	$v_0(\xi)$	$\theta_1(\xi)$	$\theta_2(\xi)$	$M_1(\xi)$	$M_2(\xi)$	$\overline{M}_1(\xi)$	$\overline{M}_2(\xi)$	$Q_1(\xi)$	$Q_2(\xi)$
0.40	1.0121	0.0382	−0.0581	0.1925	−0.2763	0.9744	−0.1464	0.4814	−0.1655	−0.1173
0.45	1.0151	0.0485	−0.0649	0.2173	−0.2676	0.9804	−0.1442	0.4828	−0.1866	−0.1312
0.50	1.0185	0.0600	−0.0703	0.2424	−0.2576	0.9872	−0.1418	0.4846	−0.2090	−0.1448
0.55	1.0223	0.0727	−0.0765	0.2675	−0.2467	0.9947	−0.1388	0.4865	−0.2309	−0.1584
0.60	1.0262	0.0868	−0.0816	0.2931	−0.2346	1.0028	−0.1362	0.4885	−0.2536	−0.1712
0.65	1.0304	0.1021	−0.0863	0.3191	−0.2214	1.0116	−0.1327	0.4907	−0.2767	−0.1838
0.70	1.0348	0.1187	−0.0905	0.3452	−0.2069	1.0208	−0.1292	0.4931	−0.3003	−0.1962
0.75	1.0393	0.1367	−0.0941	0.3718	−0.1912	1.0307	−0.1254	0.4957	−0.3245	−0.2081
0.80	1.0441	0.1559	−0.0970	0.3987	−0.1744	1.0412	−0.1213	0.4984	−0.3492	−0.2195
0.85	1.0491	0.1765	−0.0994	0.4261	−0.1563	1.0521	−0.1169	0.5012	−0.3745	−0.2304
0.90	1.0541	0.1985	−0.1010	0.4539	−0.1370	1.0639	−0.1123	0.5042	−0.4003	−0.2407
0.95	1.0591	0.2219	−0.1019	0.4821	−0.1163	1.0759	−0.1072	0.5074	−0.4270	−0.2506
1.00	1.0643	0.2466	−0.1019	0.5108	−0.0943	1.0884	−0.1018	0.5107	−0.4542	−0.2597

$\varphi = 55°$

ξ	$u_0(\xi)$	$v_0(\xi)$	$\theta_1(\xi)$	$\theta_2(\xi)$	$M_1(\xi)$	$M_2(\xi)$	$\overline{M}_1(\xi)$	$\overline{M}_2(\xi)$	$Q_1(\xi)$	$Q_2(\xi)$
0.00	1.0000	0.0000	0.0000	0.0000	−0.3420	0.9397	−0.1710	0.4698	0.0000	0.0000
0.05	1.0002	−0.0006	−0.0085	0.0234	−0.3415	0.9401	−0.1709	0.4699	−0.0191	−0.0160
0.10	1.0008	−0.0023	−0.0170	0.0470	−0.3401	0.9412	−0.1705	0.4702	−0.0383	−0.0321
0.15	1.0019	−0.0053	−0.0255	0.0706	−0.3376	0.9433	−0.1699	0.4707	−0.0586	−0.0481
0.20	1.0034	−0.0094	−0.0339	0.0942	−0.3344	0.9461	−0.1691	0.4714	−0.0770	−0.0641
0.25	1.0053	−0.0147	−0.0420	0.1181	−0.3300	0.9496	−0.1680	0.4723	−0.0965	−0.0798
0.30	1.0076	−0.0212	−0.0500	0.1420	−0.3247	0.9539	−0.1667	0.4732	−0.1164	−0.0955
0.35	1.0103	−0.0289	−0.0577	0.1662	−0.3183	0.9592	−0.1652	0.4747	−0.1304	−0.1111
0.40	1.0134	−0.0378	−0.0654	0.1905	−0.3110	0.9651	−0.1634	0.4762	−0.1566	−0.1265
0.45	1.0168	−0.0480	−0.0725	0.2151	−0.3026	0.9719	−0.1613	0.4779	−0.1773	−0.1418
0.50	1.0206	−0.0593	−0.0795	0.2407	−0.2933	0.9794	−0.1590	0.4797	−0.1981	−0.1578
0.55	1.0247	−0.0720	−0.0860	0.2651	−0.2827	0.9875	−0.1563	0.4818	−0.2197	−0.1714
0.60	1.0292	−0.0859	−0.0920	0.2904	−0.2713	0.9965	−0.1535	0.4841	−0.2414	−0.1858
0.65	1.0340	−0.1010	−0.0977	0.3162	−0.2587	1.0061	−0.1504	0.4866	−0.2637	−0.2000
0.70	1.0390	−0.1175	−0.1029	0.3424	−0.2449	1.0165	−0.1471	0.4892	−0.2865	−0.2138
0.75	1.0442	−0.1353	−0.1075	0.3690	−0.2300	1.0275	−0.1434	0.4990	−0.3100	−0.2268
0.80	1.0497	−0.1544	−0.1115	0.3960	−0.2139	1.0392	−0.1395	0.4950	−0.3339	−0.2402
0.85	1.0554	−0.1749	−0.1150	0.4234	−0.1965	1.0516	−0.1353	0.4981	−0.3585	−0.2528
0.90	1.0612	−0.1967	−0.1171	0.4513	−0.1781	1.0645	−0.1308	0.5015	−0.3838	−0.2649
0.95	1.0671	−0.2200	−0.1232	0.4826	−0.1582	1.0780	−0.1261	0.5050	−0.4110	−0.2814
1.00	1.0732	−0.2447	−0.1203	0.5087	−0.1380	1.0922	−0.1210	0.5087	−0.4366	−0.2876

$\varphi = 56°$

ξ	$u_0(\xi)$	$v_0(\xi)$	$\theta_1(\xi)$	$\theta_2(\xi)$	$M_1(\xi)$	$M_2(\xi)$	$\overline{M}_1(\xi)$	$\overline{M}_2(\xi)$	$Q_1(\xi)$	$Q_2(\xi)$
0.00	1.0000	0.0000	0.0000	0.0000	−0.3764	0.9272	−0.1873	0.4636	0.0000	0.0000
0.05	1.0002	−0.0006	−0.0094	0.0232	−0.3760	0.9276	−0.1872	0.4636	−0.0180	−0.0174
0.10	1.0009	−0.0023	−0.0188	0.0464	−0.3746	0.9289	−0.1869	0.4640	−0.0359	−0.0347
0.15	1.0021	−0.0052	−0.0281	0.0696	−0.3724	0.9311	−0.1863	0.4646	−0.0541	−0.0520
0.20	1.0037	−0.0093	−0.0373	0.0930	−0.3692	0.9341	−0.1855	0.4653	−0.0723	−0.0693
0.25	1.0058	−0.0145	−0.0463	0.1162	−0.3652	0.9380	−0.1845	0.4663	−0.0907	−0.0864
0.30	1.0083	−0.0210	−0.0553	0.1403	−0.3601	0.9428	−0.1833	0.4675	−0.1094	−0.1035
0.35	1.0113	−0.0285	−0.0640	0.1642	−0.3542	0.9484	−0.1818	0.4689	−0.1282	−0.1205
0.40	1.0147	−0.0374	−0.0724	0.1882	−0.3472	0.9548	−0.1800	0.4705	−0.1475	−0.1376
0.45	1.0185	−0.0473	−0.0806	0.2125	−0.3394	0.9721	−0.1781	0.4726	−0.1670	−0.1540
0.50	1.0227	−0.0587	−0.0884	0.2372	−0.3306	0.9702	−0.1760	0.4743	−0.1859	−0.1704
0.55	1.0273	−0.0711	−0.0959	0.2622	−0.3207	0.9791	−0.1735	0.4766	−0.2073	−0.1867
0.60	1.0323	−0.0848	−0.1030	0.2875	−0.3099	0.9889	−0.1709	0.4791	−0.2280	−0.2027
0.65	1.0376	−0.0998	−0.1097	0.3131	−0.2979	1.0095	−0.1680	0.4818	−0.2494	−0.2185
0.70	1.0428	−0.1162	−0.1159	0.3392	−0.2849	1.0108	−0.1647	0.4847	−0.2713	−0.2340
0.75	1.0491	−0.1338	−0.1215	0.3658	−0.2708	1.0328	−0.1613	0.4877	−0.2938	−0.2492
0.80	1.0552	−0.1527	−0.1268	0.3928	−0.2555	1.0356	−0.1576	0.4910	−0.3170	−0.2640
0.85	1.0617	−0.1732	−0.1314	0.4204	−0.2391	1.0592	−0.1537	0.4945	−0.3408	−0.2784
0.90	1.0682	−0.1947	−0.1352	0.4482	−0.2215	1.0635	−0.1494	0.4981	−0.3653	−0.2926
0.95	1.0751	−0.2179	−0.1385	0.4768	−0.2025	1.0884	−0.1449	0.5019	−0.3906	−0.3062
1.00	1.0820	−0.2424	−0.1409	0.5060	−0.1823	1.0942	−0.1400	0.5059	−0.4155	−0.3199

$\varphi = 57°$

ξ	$u_0(\xi)$	$v_0(\xi)$	$\theta_1(\xi)$	$\theta_2(\xi)$	$M_1(\xi)$	$M_2(\xi)$	$\overline{M}_1(\xi)$	$\overline{M}_2(\xi)$	$Q_1(\xi)$	$Q_2(\xi)$
0.00	1.0000	0.0000	0.0000	0.0000	−0.4067	0.9135	−0.2033	0.4567	0.0000	0.0000
0.05	1.0002	−0.0006	−0.0102	0.0228	−0.4063	0.9140	−0.2032	0.4568	−0.0167	−0.0186
0.10	1.0010	−0.0022	−0.0203	0.0457	−0.4050	0.9154	−0.2029	0.4572	−0.0335	−0.0373
0.15	1.0023	−0.0051	−0.0304	0.0686	−0.4030	0.9177	−0.2024	0.4577	−0.0503	−0.0558
0.20	1.0041	−0.0091	−0.0404	0.0917	−0.4000	0.9209	−0.2016	0.4585	−0.0674	−0.0743
0.25	1.0063	−0.0143	−0.0502	0.1189	−0.3963	0.9251	−0.2007	0.4596	−0.0845	−0.0928

ξ	$u_0(\xi)$	$v_0(\xi)$	$\theta_1(\xi)$	$\theta_2(\xi)$	$M_1(\xi)$	$M_2(\xi)$	$\overline{M}_1(\xi)$	$\overline{M}_2(\xi)$	$Q_1(\xi)$	$Q_2(\xi)$
0.30	1.0090	−0.0206	−0.0599	0.1382	−0.3916	0.9302	−0.1996	0.4609	−0.1019	−0.1113
0.35	1.0122	−0.0282	−0.0694	0.1619	−0.3860	0.9362	−0.1982	0.4624	−0.1195	−0.1296
0.40	1.0161	−0.0368	−0.0786	0.1857	−0.3795	0.9431	−0.1965	0.4641	−0.1376	−0.1478
0.45	1.0202	−0.0467	−0.0877	0.2097	−0.3722	0.9509	−0.1948	0.4661	−0.1559	−0.1659
0.50	1.0247	−0.0578	−0.0964	0.2242	−0.3640	0.9596	−0.1927	0.4683	−0.1746	−0.1838
0.55	1.0297	−0.0702	−0.1048	0.2589	−0.3548	0.9693	−0.1905	0.4706	−0.1939	−0.2016
0.60	1.0352	−0.0837	−0.1126	0.2840	−0.3446	0.9798	−0.1880	0.4733	−0.2135	−0.2191
0.65	1.0410	−0.0986	−0.1204	0.3095	−0.3334	0.9911	−0.1853	0.4762	−0.2337	−0.2365
0.70	1.0473	−0.1147	−0.1276	0.3355	−0.3211	1.0034	−0.1823	0.4792	−0.2546	−0.2538
0.75	1.0538	−0.1322	−0.1343	0.3620	−0.3079	1.0164	−0.1790	0.4826	−0.2761	−0.2707
0.80	1.0607	−0.1509	−0.1405	0.3889	−0.2936	1.0303	−0.1756	0.4863	−0.2982	−0.2873
0.85	1.0677	−0.1711	−0.1460	0.4164	−0.2782	1.0450	−0.1719	0.4898	−0.3210	−0.3037
0.90	1.0757	−0.1926	−0.1510	0.4444	−0.2615	1.0606	−0.1679	0.4938	−0.3447	−0.3197
0.95	1.0829	−0.2154	−0.1555	0.4731	−0.2436	1.0770	−0.1637	0.4980	−0.3690	−0.3353
1.00	1.0908	−0.2399	−0.1590	0.5023	−0.2245	1.0941	−0.1591	0.5023	−0.3942	−0.3506

$\varphi = 58°$

ξ	$u_0(\xi)$	$v_0(\xi)$	$\theta_1(\xi)$	$\theta_2(\xi)$	$M_1(\xi)$	$M_2(\xi)$	$\overline{M}_1(\xi)$	$\overline{M}_2(\xi)$	$Q_1(\xi)$	$Q_2(\xi)$
0.00	1.0000	0.0000	0.0000	0.0000	−0.4384	0.8988	−0.2192	0.4494	0.0000	0.0000
0.05	1.0003	−0.0006	−0.0110	0.0225	−0.4380	0.8993	−0.2191	0.4495	−0.0153	−0.0197
0.10	1.0010	−0.0022	−0.0219	0.0449	−0.4369	0.9007	−0.2188	0.4499	−0.0309	−0.0395
0.15	1.0025	−0.0050	−0.0328	0.0676	−0.4349	0.9032	−0.2183	0.4505	−0.0463	−0.0592
0.20	1.0044	−0.0090	−0.0435	0.0903	−0.4323	0.9067	−0.2177	0.4514	−0.0621	−0.0789
0.25	1.0068	−0.0140	−0.0542	0.1131	−0.4287	0.9111	−0.2168	0.4545	−0.0778	−0.0986
0.30	1.0098	−0.0203	−0.0648	0.1361	−0.4245	0.9165	−0.2158	0.4538	−0.0939	−0.1180
0.35	1.0133	−0.0277	−0.0751	0.1594	−0.4194	0.9229	−0.2145	0.4554	−0.1103	−0.1375
0.40	1.0173	−0.0362	−0.0852	0.1829	−0.4134	0.9302	−0.2130	0.4573	−0.1270	−0.1570
0.45	1.0218	−0.0460	−0.0951	0.2067	−0.4066	0.9386	−0.2113	0.4594	−0.1441	−0.1763
0.50	1.0268	−0.0570	−0.1047	0.2308	−0.3989	0.9478	−0.2094	0.4617	−0.1615	−0.1955
0.55	1.0322	−0.0691	−0.1141	0.2554	−0.3904	0.9581	−0.2074	0.4643	−0.1795	−0.2147
0.60	1.0382	−0.0825	−0.1265	0.2802	−0.3810	0.9693	−0.2051	0.4670	−0.1979	−0.2337
0.65	1.0446	−0.0971	−0.1316	0.3055	−0.3706	0.9814	−0.2025	0.4701	−0.2170	−0.2525
0.70	1.0514	−0.1130	−0.1398	0.3314	−0.3593	0.9945	−0.1998	0.4734	−0.2355	−0.2712
0.75	1.0585	−0.1303	−0.1476	0.3577	−0.3470	1.0085	−0.1968	0.4769	−0.2568	−0.2898
0.80	1.0661	−0.1488	−0.1549	0.3620	−0.3336	1.0235	−0.1936	0.4807	−0.2778	−0.3082
0.85	1.0741	−0.1687	−0.1617	0.4120	−0.3192	1.0392	−0.1902	0.4847	−0.2996	−0.3262
0.90	1.0823	−0.1901	−0.1679	0.4400	−0.3037	1.0561	−0.1864	0.4889	−0.3221	−0.3440
0.95	1.0908	−0.2127	−0.1734	0.4687	−0.2870	1.0737	−0.1825	0.4934	−0.3455	−0.3617
1.00	1.0996	−0.2369	−0.1784	0.4986	−0.2691	1.0922	−0.1783	0.4981	−0.3695	−0.3790

$\varphi = 59°$

ξ	$u_0(\xi)$	$v_0(\xi)$	$\theta_1(\xi)$	$\theta_2(\xi)$	$M_1(\xi)$	$M_2(\xi)$	$\overline{M}_1(\xi)$	$\overline{M}_2(\xi)$	$Q_1(\xi)$	$Q_2(\xi)$
0.00	1.0000	0.0000	0.0000	0.0000	−0.4695	0.8829	−0.2347	0.4414	0.0000	0.0000
0.05	1.0003	−0.0005	−0.0117	0.0221	−0.4692	0.8834	−0.2346	0.4415	−0.0140	−0.0207
0.10	1.0011	−0.0022	−0.0235	0.0441	−0.4681	0.8850	−0.2344	0.4419	−0.0280	−0.0414
0.15	1.0026	−0.0050	−0.0351	0.0664	−0.4664	0.8874	−0.2339	0.4426	−0.0421	−0.0622
0.20	1.0050	−0.0088	−0.0466	0.0887	−0.4639	0.8912	−0.2333	0.4435	−0.0564	−0.0829
0.25	1.0073	−0.0140	−0.0582	0.1112	−0.4607	0.8958	−0.2325	0.4446	−0.0709	−0.1035
0.30	1.0107	−0.0200	−0.0695	0.1338	−0.4568	0.9015	−0.2316	0.4461	−0.0856	−0.1241
0.35	1.0143	−0.0272	−0.0807	0.1567	−0.4522	0.9083	−0.2304	0.4477	−0.1006	−0.1448
0.40	1.0186	−0.0356	−0.0917	0.1799	−0.4467	0.9161	−0.2290	0.4497	−0.1158	−0.1654
0.45	1.0234	−0.0452	−0.1024	0.2034	−0.4406	0.9248	−0.2276	0.4519	−0.1315	−0.1859
0.50	1.0288	−0.0560	−0.1129	0.2272	−0.4336	0.9346	−0.2258	0.4543	−0.1476	−0.2063
0.55	1.0355	−0.0680	−0.1232	0.2514	−0.4258	0.9455	−0.2240	0.4571	−0.1641	−0.2268
0.60	1.0411	−0.0811	−0.1331	0.2761	−0.4172	0.9573	−0.2219	0.4600	−0.1813	−0.2471
0.65	1.0481	−0.0955	−0.1427	0.3011	−0.4076	0.9702	−0.2195	0.4633	−0.1989	−0.2673
0.70	1.0554	−0.1112	−0.1519	0.3268	−0.3973	0.9840	−0.2170	0.4667	−0.2172	−0.2875
0.75	1.0632	−0.1282	−0.1608	0.3528	−0.3800	0.9989	−0.2143	0.4704	−0.2361	−0.3075
0.80	1.0715	−0.1466	−0.1691	0.3796	−0.3738	1.0147	−0.2113	0.4744	−0.2558	−0.3275
0.85	1.0801	−0.1662	−0.1769	0.4069	−0.3603	1.0317	−0.2082	0.4787	−0.2763	−0.3472
0.90	1.0893	−0.1873	−0.1843	0.4349	−0.3460	1.0495	−0.2048	0.4832	−0.2975	−0.3668
0.95	1.0986	−0.2097	−0.1910	0.4636	−0.3306	1.0683	−0.2012	0.4879	−0.3197	−0.3863
1.00	1.1083	−0.2336	−0.1972	0.4929	−0.3140	1.0881	−0.1973	0.4929	−0.3427	−0.4056

$\varphi = 60°$

ξ	$u_0(\xi)$	$v_0(\xi)$	$\theta_1(\xi)$	$\theta_2(\xi)$	$M_1(\xi)$	$M_2(\xi)$	$\overline{M}_1(\xi)$	$\overline{M}_2(\xi)$	$Q_1(\xi)$	$Q_2(\xi)$
0.00	1.0000	0.0000	0.0000	0.0000	−0.5000	0.8660	−0.2500	0.4330	0.0000	0.0000
0.05	1.0003	−0.0005	−0.0125	0.0216	−0.4957	0.8664	−0.2499	0.4331	−0.0125	−0.0108
0.10	1.0012	−0.0022	−0.0250	0.0432	−0.4988	0.8680	−0.2497	0.4335	−0.0250	−0.0216
0.15	1.0028	−0.0049	−0.0374	0.0650	−0.4973	0.8708	−0.2493	0.4342	−0.0377	−0.0324
0.20	1.0050	−0.0087	−0.0498	0.0870	−0.4950	0.8745	−0.2488	0.4352	−0.0505	−0.0433

ξ	$u_0(\xi)$	$v_0(\xi)$	$\theta_1(\xi)$	$\theta_2(\xi)$	$M_1(\xi)$	$M_2(\xi)$	$\overline{M}_1(\xi)$	$\overline{M}_2(\xi)$	$Q_1(\xi)$	$Q_2(\xi)$
0.25	1.0078	−0.0136	−0.0620	0.1090	−0.4921	0.8793	−0.2481	0.4364	−0.0634	−0.0541
0.30	1.0112	−0.0196	−0.0741	0.1312	−0.4886	0.8853	−0.2472	0.4379	−0.0766	−0.0649
0.35	1.0152	−0.0267	−0.0862	0.1537	−0.4839	0.8924	−0.2462	0.4396	−0.0901	−0.0757
0.40	1.0198	−0.0350	−0.0979	0.1765	−0.4796	0.9005	−0.2449	0.4416	−0.1040	−0.0865
0.45	1.0250	−0.0444	−0.1096	0.1998	−0.4740	0.9097	−0.2436	0.4439	−0.1182	−0.0974
0.50	1.0307	−0.0550	−0.1210	0.2232	−0.4678	0.9200	−0.2420	0.4465	−0.1328	−0.1082
0.55	1.0371	−0.0667	−0.1322	0.2471	−0.4607	0.9314	−0.2404	0.4494	−0.1479	−0.1190
0.60	1.0440	−0.0797	−0.1430	0.2768	−0.4530	0.9438	−0.2385	0.4525	−0.1636	−0.1298
0.65	1.0514	−0.0939	−0.1535	0.2962	−0.4444	0.9573	−0.2365	0.4558	−0.1798	−0.1405
0.70	1.0593	−0.1093	−0.1633	0.3215	−0.4350	0.9719	−0.2341	0.4595	−0.1966	−0.1513
0.75	1.0678	−0.1261	−0.1736	0.3476	−0.4237	0.9876	−0.2317	0.4634	−0.2142	−0.1621
0.80	1.0767	−0.1441	−0.1832	0.3741	−0.4135	1.0043	−0.2289	0.4676	−0.2324	−0.1728
0.85	1.0861	−0.1635	−0.1921	0.4012	−0.4015	1.0222	−0.2261	0.4721	−0.2514	−0.1835
0.90	1.0959	−0.1842	−0.2006	0.4290	−0.3884	1.0410	−0.2230	0.4768	−0.2713	−0.1941
0.95	1.1061	−0.2064	−0.2086	0.4576	−0.3743	1.0611	−0.2198	0.4818	−0.2920	−0.2047
1.00	1.1167	−0.2300	−0.2161	0.4870	−0.3591	1.0821	−0.2162	0.4871	−0.3138	−0.2153
					$\varphi = 61°$					
0.00	1.0000	0.0000	0.0000	0.0000	−0.5299	0.8480	−0.2649	0.4240	0.0000	0.0000
0.05	1.0003	−0.0005	−0.0132	0.0218	−0.5296	0.8486	−0.2648	0.4241	−0.0109	−0.0225
0.10	1.0013	−0.0021	−0.0265	0.0425	−0.5288	0.8502	−0.2646	0.4246	−0.0220	−0.0449
0.15	1.0030	−0.0048	−0.0396	0.0638	−0.5274	0.8531	−0.2643	0.4253	−0.0331	−0.0674
0.20	1.0053	−0.0085	−0.0528	0.0852	−0.5255	0.8570	−0.2638	0.4262	−0.0443	−0.0899
0.25	1.0083	−0.0132	−0.0658	0.1069	−0.5230	0.8620	−0.2632	0.4275	−0.0558	−0.1124
0.30	1.0119	−0.0192	−0.0788	0.1287	−0.5199	0.8682	−0.2625	0.4290	−0.0675	−0.1350
0.35	1.0161	−0.0262	−0.0915	0.1508	−0.5163	0.8755	−0.2616	0.4309	−0.0794	−0.1576
0.40	1.0210	−0.0343	−0.1042	0.1732	−0.5120	0.8840	−0.2604	0.4330	−0.0917	−0.1802
0.45	1.0265	−0.0435	−0.1167	0.1959	−0.5071	0.8936	−0.2593	0.4354	−0.1043	−0.2028
0.50	1.0327	−0.0539	−0.1290	0.2170	−0.5015	0.9043	−0.2579	0.4380	−0.1174	−0.2254
0.55	1.0395	−0.0654	−0.1410	0.2425	−0.4954	0.9161	−0.2565	0.4410	−0.1309	−0.2482
0.60	1.0468	−0.0781	−0.1529	0.2665	−0.4885	0.9292	−0.2548	0.4442	−0.1450	−0.2708
0.65	1.0547	−0.0921	−0.1644	0.2910	−0.4808	0.9433	−0.2529	0.4477	−0.1597	−0.2937
0.70	1.0633	−0.1073	−0.1757	0.3161	−0.4725	0.9586	−0.2509	0.4516	−0.1750	−0.3165
0.75	1.0722	−0.1236	−0.1865	0.3418	−0.4634	0.9749	−0.2488	0.4557	−0.1930	−0.3393
0.80	1.0819	−0.1414	−0.1972	0.3681	−0.4535	0.9926	−0.2463	0.4600	−0.2077	−0.3621
0.85	1.0919	−0.1605	−0.2073	0.3950	−0.4425	1.0112	−0.2439	0.4647	−0.2262	−0.3851
0.90	1.1026	−0.1809	−0.2170	0.4227	−0.4308	1.0311	−0.2411	0.4697	−0.2435	−0.4081
0.95	1.1136	−0.2027	−0.2260	0.4511	−0.4182	1.0521	−0.2382	0.4749	−0.2617	−0.4309
1.00	1.1252	−0.2260	−0.2349	0.4805	−0.4045	1.0742	−0.2350	0.4805	−0.2829	−0.4539
					$\varphi = 62°$					
0.00	1.0000	0.0000	0.0000	0.0000	−0.5592	0.8290	−0.2796	0.4145	0.0000	0.0000
0.05	1.0003	−0.0005	−0.0132	0.0207	−0.5590	0.8296	−0.2795	0.4146	−0.0094	−0.0232
0.10	1.0013	−0.0020	−0.0265	0.0415	−0.5583	0.8313	−0.2794	0.4151	−0.0188	−0.0464
0.15	1.0030	−0.0046	−0.0396	0.0624	−0.5571	0.8342	−0.2791	0.4158	−0.0283	−0.0695
0.20	1.0053	−0.0083	−0.0527	0.0834	−0.5555	0.8383	−0.2787	0.4168	−0.0380	−0.0928
0.25	1.0083	−0.0129	−0.0658	0.1045	−0.5533	0.8435	−0.2782	0.4181	−0.0477	−0.1161
0.30	1.0119	−0.0187	−0.0788	0.1259	−0.5507	0.8499	−0.2775	0.4197	−0.0578	−0.1394
0.35	1.0161	−0.0256	−0.0916	0.1476	−0.5475	0.8574	−0.2768	0.4216	−0.0682	−0.1629
0.40	1.0211	−0.0336	−0.1043	0.1695	−0.5438	0.8662	−0.2758	0.4238	−0.0788	−0.1862
0.45	1.0266	−0.0426	−0.1169	0.1918	−0.5396	0.8760	−0.2748	0.4262	−0.0899	−0.2098
0.50	1.0327	−0.0527	−0.1363	0.2144	−0.5348	0.8871	−0.2736	0.4290	−0.1013	−0.2334
0.55	1.0395	−0.0640	−0.1415	0.2376	−0.5295	0.8994	−0.2724	0.4320	−0.1133	−0.2571
0.60	1.0468	−0.0765	−0.1535	0.2612	−0.5235	0.9128	−0.2709	0.4355	−0.1257	−0.2809
0.65	1.0549	−0.0902	−0.1653	0.2854	−0.5169	0.9275	−0.2693	0.4391	−0.1387	−0.3047
0.70	1.0634	−0.1050	−0.1768	0.3103	−0.5096	0.9434	−0.2676	0.4430	−0.1524	−0.3287
0.75	1.0725	−0.1212	−0.1881	0.3354	−0.5017	0.9604	−0.2657	0.4473	−0.1667	−0.3528
0.80	1.0822	−0.1385	−0.1989	0.3615	−0.4931	0.9785	−0.2637	0.4518	−0.1817	−0.3770
0.85	1.0924	−0.1572	−0.2094	0.3881	−0.4835	0.9981	−0.2615	0.4567	−0.1976	−0.4013
0.90	1.1031	−0.1774	−0.2195	0.4155	−0.4732	1.0186	−0.2590	0.4618	−0.2163	−0.4256
0.95	1.1143	−0.1987	−0.2293	0.4439	−0.4621	1.0406	−0.2565	0.4672	−0.2318	−0.4501
1.00	1.1261	−0.2216	−0.2387	0.4729	−0.4500	1.0637	−0.2538	0.4729	−0.2503	−0.4748
					$\varphi = 63°$					
0.00	1.0000	0.0000	0.0000	0.0000	−0.5878	0.8090	−0.2939	0.4045	0.0000	0.0000
0.05	1.0004	−0.0005	−0.0147	0.0202	−0.5876	0.8096	−0.2939	0.4046	−0.0077	−0.0228
0.10	1.0014	−0.0020	−0.0294	0.0405	−0.5870	0.8114	−0.2937	0.4051	−0.0156	−0.0475
0.15	1.0033	−0.0045	−0.0440	0.0609	−0.5861	0.8143	−0.2935	0.4058	−0.0234	−0.0714
0.20	1.0059	−0.0081	−0.0586	0.0814	−0.5847	0.8185	−0.2931	0.4069	−0.0315	−0.0953
0.25	1.0092	−0.0126	−0.0732	0.1020	−0.5830	0.8238	−0.2927	0.4082	−0.0396	−0.1192

ξ	$u_0(\xi)$	$v_0(\xi)$	$\theta_1(\xi)$	$\theta_2(\xi)$	$M_1(\xi)$	$M_2(\xi)$	$\overline{M}_1(\xi)$	$\overline{M}_2(\xi)$	$Q_1(\xi)$	$Q_2(\xi)$
0.30	1.0132	−0.0183	−0.0877	0.1229	−0.5808	0.8304	−0.2922	0.4098	−0.0480	−0.1432
0.35	1.0179	−0.0250	−0.1021	0.1441	−0.5783	0.8382	−0.2916	0.4118	−0.0567	−0.1672
0.40	1.0234	−0.0328	−0.1162	0.1656	−0.5754	0.8471	−0.2907	0.4140	−0.0657	−0.1924
0.45	1.0295	−0.0415	−0.1305	0.1874	−0.5720	0.8573	−0.2899	0.4165	−0.0751	−0.2158
0.50	1.0364	−0.0515	−0.1414	0.2096	−0.5684	0.8687	−0.2889	0.4194	−0.0849	−0.2402
0.55	1.0440	−0.0625	−0.1583	0.2324	−0.5643	0.8813	−0.2879	0.4226	−0.0951	−0.2646
0.60	1.0523	−0.0747	−0.1719	0.2555	−0.5597	0.8952	−0.2867	0.4260	−0.1059	−0.2894
0.65	1.0612	−0.0880	−0.1853	0.2793	−0.5548	0.9103	−0.2854	0.4297	−0.1171	−0.3143
0.70	1.0709	−0.1072	−0.1987	0.3036	−0.5495	0.9266	−0.2839	0.4338	−0.1291	−0.3392
0.75	1.0810	−0.1185	−0.2117	0.3287	−0.5437	0.9442	−0.2823	0.4381	−0.1417	−0.3644
0.80	1.0919	−0.1355	−0.2244	0.3543	−0.5376	0.9681	−0.2806	0.4428	−0.1550	−0.3898
0.85	1.1035	−0.1538	−0.2368	0.3807	−0.5310	0.9882	−0.2787	0.4478	−0.1691	−0.4154
0.90	1.1156	−0.1735	−0.2489	0.4078	−0.5239	1.0047	−0.2767	0.4531	−0.1839	−0.4412
0.95	1.1284	−0.1945	−0.2607	0.4359	−0.5165	1.0273	−0.2746	0.4587	−0.1997	−0.4672
1.00	1.1417	−0.2169	−0.2721	0.4647	−0.5085	1.0513	−0.2722	0.4647	−0.2163	−0.4933
					$\varphi = 64°$					
0.00	1.0000	−0.0000	−0.0000	0.0000	−0.6157	0.7880	−0.3078	0.3940	0.0000	0.0000
0.05	1.0004	−0.0005	−0.0154	0.0197	−0.6156	0.7886	−0.3078	0.3941	−0.0060	−0.0242
0.10	1.0015	−0.0019	−0.0308	0.0395	−0.6145	0.7904	−0.3077	0.3946	−0.0121	−0.0485
0.15	1.0035	−0.0044	−0.0461	0.0593	−0.6143	0.7934	−0.3075	0.3954	−0.0183	−0.0729
0.20	1.0061	−0.0079	−0.0615	0.0793	−0.6133	0.7977	−0.3072	0.3964	−0.0246	−0.0972
0.25	1.0096	−0.0123	−0.0768	0.0994	−0.6119	0.8032	−0.3069	0.3978	−0.0310	−0.1217
0.30	1.0138	−0.0178	−0.0919	0.1198	−0.6103	0.8098	−0.3065	0.3994	−0.0378	−0.1662
0.35	1.0188	−0.0243	−0.1071	0.1405	−0.6083	0.8178	−0.3060	0.4014	−0.0447	−0.1709
0.40	1.0245	−0.0319	−0.1221	0.1615	−0.6060	0.8270	−0.3054	0.4037	−0.0520	−0.1957
0.45	1.0310	−0.0405	−0.1371	0.1828	−0.6034	0.8374	−0.3047	0.4063	−0.0596	−0.2206
0.50	1.0383	−0.0501	−0.1519	0.2046	−0.6006	0.8490	−0.3040	0.4093	−0.0677	−0.2458
0.55	1.0463	−0.0610	−0.1667	0.2268	−0.5974	0.8620	−0.3031	0.4124	−0.0761	−0.2709
0.60	1.0549	−0.0729	−0.1812	0.2496	−0.5937	0.8761	−0.3021	0.4159	−0.0851	−0.2965
0.65	1.0643	−0.0859	−0.1956	0.2728	−0.5899	0.8916	−0.3011	0.4198	−0.0946	−0.3222
0.70	1.0745	−0.1001	−0.2099	0.2968	−0.5858	0.9083	−0.2999	0.4239	−0.1047	−0.3481
0.75	1.0853	−0.1156	−0.2239	0.3213	−0.5811	0.9254	−0.2986	0.4284	−0.1153	−0.3743
0.80	1.0969	−0.1323	−0.2379	0.3465	−0.5763	0.9458	−0.2972	0.4332	−0.1248	−0.4007
0.85	1.1091	−0.1502	−0.2513	0.3725	−0.5712	0.9664	−0.2958	0.4383	−0.1389	−0.4275
0.90	1.1220	−0.1692	−0.2647	0.3994	−0.5656	0.9886	−0.2940	0.4438	−0.1519	−0.4544
0.95	1.1355	−0.1900	−0.2776	0.4271	−0.5596	1.0120	−0.2926	0.4496	−0.1657	−0.4818
1.00	1.1498	−0.2119	−0.2903	0.4556	−0.5533	1.0367	−0.2906	0.4557	−0.1802	−0.5093
					$\varphi = 65°$					
0.00	1.0000	0.0000	0.0000	0.0000	−0.2588	0.7660	−0.3214	0.3830	0.0000	0.0000
0.05	1.0004	−0.0005	−0.0161	−0.0190	−0.2586	0.7665	−0.3214	0.3831	−0.0043	−0.0245
0.10	1.0016	−0.0019	−0.0322	−0.0382	−0.2578	0.7684	−0.3213	0.3836	−0.0087	−0.0493
0.15	1.0036	−0.0043	−0.0481	−0.0576	−0.2565	0.7715	−0.3212	0.3844	−0.0132	−0.0749
0.20	1.0064	−0.0077	−0.0659	−0.0770	−0.2546	0.7758	−0.3210	0.3855	−0.0177	−0.0996
0.25	1.0100	−0.0120	−0.0818	−0.0974	−0.2523	0.7814	−0.3207	0.3868	−0.0221	−0.1262
0.30	1.0144	−0.0174	−0.0961	−0.1165	−0.2493	0.7881	−0.3204	0.3885	−0.0257	−0.1495
0.35	1.0196	−0.0237	−0.1121	−0.1366	−0.2458	0.7962	−0.3201	0.3905	−0.0327	−0.1736
0.40	1.0256	−0.0310	−0.1279	−0.1571	−0.2417	0.8055	−0.3196	0.3928	−0.0356	−0.1990
0.45	1.0324	−0.0394	−0.1436	−0.1780	−0.2371	0.8161	−0.3191	0.3955	−0.0415	−0.2244
0.50	1.0340	−0.0488	−0.1593	−0.2022	−0.2317	0.8280	−0.3186	0.3985	−0.0476	−0.2500
0.55	1.0483	−0.0593	−0.1748	−0.2209	−0.2259	0.8411	−0.3179	0.4017	−0.0544	−0.2759
0.60	1.0575	−0.0709	−0.1903	−0.2433	−0.2193	0.8556	−0.3172	0.4054	−0.0614	−0.3021
0.65	1.0674	−0.0837	−0.2056	−0.2660	−0.2122	0.8712	−0.3164	0.4092	−0.0690	−0.3285
0.70	1.0780	−0.0976	−0.2209	−0.2895	−0.2043	0.8884	−0.3156	0.4135	−0.0772	−0.3552
0.75	1.0895	−0.1126	−0.2359	−0.3134	−0.1958	0.9068	−0.3146	0.4180	−0.0859	−0.3822
0.80	1.1016	−0.1289	−0.2508	−0.3383	−0.1865	0.9266	−0.3136	0.4229	−0.0954	−0.4096
0.85	1.1145	−0.1465	−0.2655	−0.3639	−0.1764	0.9478	−0.3124	0.4282	−0.1056	−0.4372
0.90	1.1282	−0.1653	−0.2800	−0.3895	−0.1654	0.9703	−0.3111	0.4337	−0.1165	−0.4653
0.95	1.1425	−0.1855	−0.2943	−0.4176	−0.1537	0.9943	−0.3096	0.4395	−0.1281	−0.4939
1.00	1.1576	−0.2071	−0.3083	−0.4458	−0.1410	1.0198	−0.3084	0.4455	−0.1434	−0.5226

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